Variance and Covariance Analyses for Unbalanced Classifications

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Introduction

Yates [20,21], Snedecor [16], Snedecor and Cox [17], Nair [12], Cochran [3], Stevens [18], Henderson [10], and Rao [15] have presented analysis of variance procedures for unbalanced classifications while Day and Fisher [5], Wilks [19], Hazel [9], Rao [15], Henderson [11], Das [4], and Federer [6,7,8] have discussed covariance analyses for unbalanced classifications. Bartlett [1,2], Quenouille [14], Federer [7] and Outhwaite and Rutherford [13], among others, have discussed the use of dummy covariates to remove the effect of disproportion in particular unbalanced classifications.

The purpose of this paper is to present additional results for variance and covariance analyses with unbalanced classifications. In particular, the analyses are grouped as follows:

Case I -- Interaction absent;

Case II -- Interaction present; the effects assumed to be fixed effects;

Case III -- Interaction present; the interaction effects and at least one of the main effects of the factors represented in the interactions assumed to be random effects.

A Case I variance analysis is known as "the method of fitting constants" and a Case II variance analysis is known as "the weighted squares of means analysis" in the literature [16,17,20,21]. A Case III analysis of variance has been discussed in the literature for balanced classifications (e.g., see 7, ch. VIII), but not for unbalanced classifications. A covariance analysis

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for one-way classifications has been discussed by Day and Fisher [5]. Case I covariance analyses for unbalanced classifications have been presented by Das [4], Federer [6,8], and Hazel [9]. Federer [6,8] presented the Case II and the Case III linear covariance analyses for an unbalanced two-way classification, and included illustrative numerical examples. The additional results included in the present paper represent extensions of previous work [4,6,8] to unbalanced q-way classifications with several covariates.

Case I

The linear model for a two-way classification (Table 1) with a covariate is:

$$Y_{ijh} = \mu + \tau_i + \rho_j + \epsilon_{ijh} + \beta(X_{ijh} - \bar{x}) , \qquad \text{(I-1)}$$

where μ = an effect common to all observations, τ_i = an effect common to the ith level of the first factor or classification, ρ_j = an effect common to the jth level of the second factor or classification, ϵ_{ijh} = a random error component, $\bar{x}$ = arithmetic mean of the X's, and β = the regression of Y on X in the residual line of the analysis of covariance (Table 2). The τ_i and ρ_j are considered to be fixed effects. With multiple regression the linear model becomes:

$$Y_{ijh} = \mu + \tau_i + \rho_j + \epsilon_{ijh} + \sum_{g=1}^{b} \beta_g(X_{gijh} - \bar{x}_g) \qquad \text{(I-2)}$$

where β_g is the partial regression of Y on the gth independent variate, X_g, in the residual line of the analysis of covariance and $\bar{x}_g$ is the arithmetic mean of the X's in the gth group.

The normal equations obtained for the linear model given by formula (I-2) are:

<u>Equation for $\hat{\mu}$:</u>

$$\sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{n_{ij}} Y_{ijh} = Y_{...} = n_{..}\hat{\mu} + \sum_{i=1}^{v} n_{i.}\hat{\tau}_i + \sum_{j=1}^{r} n_{.j}\hat{\rho}_j \quad ; \qquad \text{(I-3)}$$

<u>Equations for $\hat{\tau}_i$:</u>

$$\sum_{j=1}^{r} \sum_{h=1}^{n_{ij}} Y_{ijh} = Y_{i..} = n_{i.}(\hat{\mu} + \hat{\tau}_i) + \sum_{j=1}^{r} n_{ij}\hat{\rho}_j + \sum_{g=1}^{b} \hat{\beta}_g(X_{gi..} - n_{i.}\bar{x}_g) \quad ; \qquad \text{(I-4)}$$

<u>Equations for $\hat{\rho}_j$:</u>

$$\sum_{i=1}^{v} \sum_{h=1}^{n_{ij}} Y_{.j.} = n_{.j}(\hat{\mu} + \hat{\rho}_j) + \sum_{i=1}^{v} n_{ij}\hat{\tau}_i + \sum_{g=1}^{b} \hat{\beta}_g(X_{g.j.} - n_{.j}\bar{x}_g) \quad ; \qquad \text{(I-5)}$$

Table 1. Yield and number of observations for the variate Y and for the covariate X from a two-way classification (totals per n_{ij} observations).

first category	second category						Total
	1	2	...	j	...	r	
1	n_{11} $Y_{11.}$ $X_{11.}$	n_{12} $Y_{12.}$ $X_{12.}$		n_{1j} $Y_{1j.}$ $X_{1j.}$		n_{1r} $Y_{1r.}$ $X_{1r.}$	$n_{1.}$ $Y_{1..}$ $X_{1..}$
2	n_{21} $Y_{21.}$ $X_{21.}$	n_{22} $Y_{22.}$ $X_{22.}$		n_{2j} $Y_{2j.}$ $X_{2j.}$		n_{2r} $Y_{2r.}$ $X_{2r.}$	$n_{2.}$ $Y_{2..}$ $X_{2..}$
⋮							
i	n_{i1} $Y_{i1.}$ $X_{i1.}$	n_{i2} $Y_{i2.}$ $X_{i2.}$		n_{ij} $Y_{ij.}$ $X_{ij.}$		n_{ir} $Y_{ir.}$ $X_{ir.}$	$n_{i.}$ $Y_{i..}$ $X_{i..}$
⋮							
v	n_{v1} $Y_{v1.}$ $X_{v1.}$	n_{v2} $Y_{v2.}$ $X_{v2.}$		n_{vj} $Y_{vj.}$ $X_{vj.}$		n_{vr} $Y_{vr.}$ $X_{vr.}$	$n_{v.}$ $Y_{v..}$ $X_{v..}$
Total	$n_{.1}$ $Y_{.1.}$ $X_{.1.}$	$n_{.2}$ $Y_{.2.}$ $X_{.2.}$		$n_{.j}$ $Y_{.j.}$ $X_{.j.}$		$n_{.r}$ $Y_{.r.}$ $X_{.r.}$	$n_{..}$ $Y_{...}$ $X_{...}$

Equations for $\hat{\beta}_g$:

$$\sum_i \sum_j \sum_h Y_{ijh}(X_{gijh} - \bar{x}_g) = \sum_{i=1}^{v} \hat{\tau}_i (X_{gi\cdot\cdot} - n_{i\cdot}\bar{x}_g) + \sum_{j=1}^{r} \hat{\rho}_j (X_{g\cdot j\cdot} - n_{\cdot j}\bar{x}_g)$$

$$+ \sum_i \sum_j \sum_h (X_{gijh} - \bar{x}_g) \sum_{g=1}^{b} \hat{\beta}_g (X_{gijh} - \bar{x}_g) \quad . \tag{I-6}$$

In the above equations $n_{i\cdot} = \sum_{j=1}^{r} n_{ij}$, $n_{\cdot j} = \sum_{i=1}^{v} n_{ij}$, $n_{\cdot\cdot} = \sum_{i=1}^{v} \sum_{j=1}^{r} n_{ij}$ and $n_{ij} = 0,1,2,\cdots$ = number of individuals in the ijth subclass. Additional restraints on the above set of equations which allow for unique solutions of the unknowns and which represent reasonable restraints are:

$$\sum_{j=1}^{r} \hat{\rho}_j = 0 \quad ; \tag{I-7}$$

$$\sum_{i=1}^{v} \hat{\tau}_i = 0 \quad . \tag{I-8}$$

The 1 + v + r + b normal equations plus equations (I-7) and (I-8) in the unknowns may be reduced to v + b equations involving only the unknowns $\hat{\tau}_i$ and $\hat{\beta}_g$. Further reduction of the v + b equations to v equations involving only the $\hat{\tau}_i$ and the observations is possible, but not too desirable computationally. The $\hat{\beta}_g$ values may be obtained from an analysis of covariance table similar to Table 2 [see 16 , section 13.7], and then a set of v equations in $\hat{\tau}_i$ where $-\hat{\tau}_1 = \sum_{i=2}^{v} \hat{\tau}_i$ is substituted in the vth equation, may be obtained as follows:

$$\begin{pmatrix} n_{1\cdot}-\Sigma\frac{n_{1j}^2}{n_{\cdot j}} & -\Sigma\frac{n_{1j}n_{2j}}{n_{\cdot j}} & \cdots & -\Sigma\frac{n_{1j}n_{vj}}{n_{\cdot j}} \\ -\Sigma\frac{n_{1j}n_{2j}}{n_{\cdot j}} & n_{2\cdot}-\Sigma\frac{n_{2j}^2}{n_{\cdot j}} & \cdots & -\Sigma\frac{n_{2j}n_{vj}}{n_{\cdot j}} \\ \vdots & & \vdots & \\ -\Sigma\frac{n_{1j}n_{v-1,j}}{n_{\cdot j}} & -\Sigma\frac{n_{2j}n_{v-1,j}}{n_{\cdot j}} & \cdots & -\Sigma\frac{n_{v-1,j}n_{vj}}{n_{\cdot j}} \\ 0 & \Sigma n_{vj}\frac{(n_{1j}-n_{2j})}{n_{\cdot j}} & \cdots & n_{v\cdot}+\Sigma\frac{n_{vj}}{n_{\cdot j}}(n_{1j}-n_{vj}) \end{pmatrix} \begin{pmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_{v-1} \\ \hat{\tau}_v \end{pmatrix} =$$

$$\begin{pmatrix} Y_{1\cdot\cdot} - \sum_j n_{1j}\bar{y}_{\cdot j\cdot} - \sum_g \hat{\beta}_g\,[X_{g1\cdot\cdot}-n_{1\cdot}\bar{x}_g-\sum_j n_{1j}(\bar{x}_{g\cdot j\cdot}-\bar{x}_g)] = Q_{1\cdot} \\ Y_{2\cdot\cdot} - \sum_j n_{2j}\bar{y}_{\cdot j\cdot} - \sum_g \hat{\beta}_g\,[X_{g2\cdot\cdot}-n_{2\cdot}\bar{x}_g-\sum_j n_{2j}(\bar{x}_{g\cdot j\cdot}-\bar{x}_g)] = Q_{2\cdot} \\ \vdots \qquad\qquad \vdots \\ Y_{v-1\cdot\cdot} - \sum_j n_{v-1,j}\bar{y}_{\cdot j\cdot} - \sum_g \hat{\beta}_g\,[X_{gv-1\cdot\cdot}-n_{v-1\cdot}\bar{x}_g-\sum_j n_{v-1,j}(\bar{x}_{g\cdot j\cdot}-\bar{x}_g)] = Q_{v-1\cdot} \\ Y_{v\cdot\cdot} - \sum_j n_{vj}\bar{y}_{\cdot j\cdot} - \sum_g \hat{\beta}_g\,[X_{gv\cdot\cdot}-n_{v\cdot}\bar{x}_g-\sum_j n_{vj}(\bar{x}_{g\cdot j\cdot}-\bar{x}_g)] = Q_{v\cdot} \end{pmatrix} \qquad (I-9)$$

The v equations from (I-9) yield unique values for the $\hat{\tau}_i$. With solutions for the $\hat{\tau}_i$, $\hat{\mu}$ and $\hat{\rho}_j$ may now be evaluated; the adjusted means are $\hat{\mu} + \hat{\tau}_i$ and $\hat{\mu} + \hat{\rho}_j$ for the first and second classifications respectively.

In the event that formula (I-1) is appropriate, an estimate of β is:

$$\hat{\beta} = \frac{\sum\sum\sum_{ijh} Y_{ijh}X_{ijh} - \sum_j Y_{\cdot j\cdot}\bar{x}_{\cdot j\cdot} - \sum_i \hat{\tau}_i(X_{i\cdot\cdot}-n_{i\cdot}\bar{x}-\sum_j n_{ij}(\bar{x}_{\cdot j\cdot}-\bar{x}))}{\sum\sum\sum_{ijh} X_{ijh}^2 - \Sigma X_{\cdot j\cdot}^2/n_{\cdot j} = W_{xx}} \qquad (I-10)$$

Likewise, the kth equation of the v equations in $\hat{\tau}_i$ is:

$$n_{k.}\hat{\tau}_k - \sum_j \frac{n_{kj}}{n_{.j}} \sum_i n_{ij}\hat{\tau}_i - \frac{X_{k..} - n_{k.}\bar{x} - \sum_j n_{kj}(\bar{x}_{.j.} - \bar{x})}{W_{xx}} \sum_i \hat{\tau}_i (X_{i..} - n_{i.}\bar{x} - \sum_j n_{ij}(\bar{x}_{.j.} - \bar{x})$$

$$= Y_{k..} - \sum_j n_{kj}\bar{y}_{.j.} - \frac{(X_{k..} - n_{k.}\bar{x} - \sum_j n_{kj}(\bar{x}_{.j.} - \bar{x})}{W_{xx}} (\Sigma\Sigma\Sigma Y_{ijh}X_{ijh} - \sum_j Y_{.j.}\bar{x}_{.j.}) \quad . \tag{I-11}$$

The above v equations plus equation (I-8) result in unique solutions for the $\hat{\tau}_i$. The estimate of $\hat{\beta}$ is then obtained from formula (I-10) or $\hat{\beta}$ may be obtained directly from Table 2 as $\hat{\beta} = D_{xy}/D_{xx}$.

Also, if the linear model in (I-1) is the correct one instead of (I-2), the right hand side of equation (I-9) is altered in that the summation is over the single term g=1. If covariance is not being considered, then each $X_{gijh}-\bar{x}_g$ is set equal to zero in equations (I-1) to (I-9). In obtaining the sums of squares given in Table 2, analyses of variance are obtained for the X_g and the Y variates and their cross products. Estimates of μ, τ_i and ρ_j. obtained from equations (I-1) to (I-9) with each $(X_{gijh}-\bar{x}_g)$ set equal to zero will be denoted as μ', ρ_j', and τ_i'. Estimates of μ and ρ_j obtained from equations (I-2), (I-3), (I-5), and (I-7) when each $\hat{\tau}_i$ and each $(X_{gijh}-\bar{x}_g)$ are set equal to zero will be denoted as μ'' and ρ_j''. Likewise, the estimate of μ and τ_i obtained from equations (I-2), (I-3), (I-4), and (I-8) when each $\hat{\rho}_j$ and each $(X_{gijh}-\bar{x}_g)$ are set equal to zero will be designated as μ^* and τ_i^*. The estimate of μ obtained from (I-2) and (I-3) when each $\hat{\tau}_i$, each $\hat{\rho}_j$, and each $(X_{gijh}-\bar{x}_g)$ are set equal to zero is $\mu'''=\bar{y}$.

Table 2. Analysis of covariance table for a two-way classification with unequal numbers in the subclasses -- Case I.

Source of variation	d.f.	Cross products y^2	xy	x^2
Total (corrected for mean)	$n_{..}-1$	T_{yy}	T_{xy}	T_{xx}
Second factor (ignoring 1st factor)	$r-1$	R_{yy}	R_{xy}	R_{xx}
First factor (eliminating 2nd factor)	$v-1$	V_{yy}	V_{xy}	V_{xx}
Residual	$n_{..}-r-v+1$	D_{yy}	D_{xy}	D_{xx}
Second factor (eliminating 1st factor)	$r-1$	B_{yy}	B_{xy}	B_{xx}

	Adjusted sums of squares/1 d.f.	ss
Residual	$n_{..}-r-v$	$D'_{yy}=D_{yy}-D^2_{xy}/D_{xx}$
First factor + residual	$n_{..}-r-1$	$W'_{yy}=W_{yy}-W^2_{xy}/W_{xx}$
First factor (eliminating regression and second factor)	$v-1$	$V'_{yy}=W'_{yy}-D'_{yy}$
Second factor + residual	$n_{..}-v-1$	$U'_{yy}=U_{yy}-U^2_{xy}/U_{xx}$
Second factor (eliminating regression and first factor)	$r-1$	$B'_{yy}=U'_{yy}-D'_{yy}$

/1 $W_{yy}=V_{yy}+D_{yy}$; $W_{xy}=V_{xy}+D_{xy}$; $W_{xx}=V_{xx}+D_{xx}$.

$U_{yy}=B_{yy}+D_{yy}$; $U_{xy}=B_{xy}+D_{xy}$; $U_{xx}=B_{xx}+D_{xx}$.

In computing the sums of squares for the variance analysis, the following procedure is used:

<u>Total corrected for the mean with $n_{..}-1$ d.f.:</u>

Total sum of squares - $SS(\hat{\hat{\hat{\mu}}}''')$ =

$$= \sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{n_{ij}} Y_{ijh}^2 - Y_{...}^2/n_{..} = T_{yy} . \qquad \text{(I-12)}$$

<u>Second category (elminating mean and ignoring first category) with r-1 d.f.:</u>/1

$SS(\mu'', \rho_j'') - SS(\hat{\hat{\hat{\mu}}}''')$

$$= (\mu'' - \hat{\hat{\hat{\mu}}}''') Y_{...} + \Sigma \rho_j'' Y_{.j.} = \sum_j \frac{Y_{.j.}^2}{n_{.j}} - \frac{Y_{...}^2}{n_{..}} = R_{yy} . \qquad \text{(I-13)}$$

<u>First category (eliminating mean and ignoring second category) with v-1 d.f.:</u>

$SS(\mu^*, \tau_i^*) - SS(\hat{\hat{\hat{\mu}}}''')$

$$= (\mu^* - \hat{\hat{\hat{\mu}}}''') Y_{...} + \Sigma \tau_i^* Y_{i..} = \sum_i \frac{Y_{i..}^2}{n_{i.}} - \frac{Y_{...}^2}{n_{..}} = A_{yy} . \qquad \text{(I-14)}$$

<u>First category (eliminating mean and second category) with v-1 d.f.:</u>

$SS(\hat{\mu}', \hat{\tau}_i', \hat{\rho}_j') - SS(\mu'', \rho_j'')$

$$= (\hat{\mu}' - \mu'') Y_{...} + \sum_j (\hat{\rho}_j' - \rho_j'') Y_{.j.} + \Sigma \hat{\tau}_i' Y_{i..}$$

$$= \hat{\mu}' Y_{...} + \sum_j \hat{\rho}_j' Y_{.j.} + \Sigma \hat{\tau}_i' Y_{i..} - \sum_j \frac{Y_{.j.}^2}{n_{.j}}$$

$$= \sum_{i=1}^{v} \hat{\tau}_i' (Y_{i..} - \sum_{j=1}^{r} n_{ij} \bar{y}_{.j.}) = V_{yy} . \qquad \text{(I-15)}$$

/1 $SS(\mu'', \rho_j'')$ is the sum of squares attributable to μ'' and the ρ_j''; etc.

Second category (eliminating mean and first category) with r-1 d.f.:

$$SS(\mu', \rho_j', \tau_i') - SS(\mu^*, \tau_i^*)$$

$$= \mu' Y_{...} + \Sigma \rho_j' Y_{.j.} + \Sigma \tau_i' Y_{i..} - \Sigma Y_{i..}^2 / n_{i.}$$

$$= \sum_{j=1}^{r} \rho_j' (Y_{.j.} - \sum_{i=1}^{v} n_{ij} \tilde{y}_{i..}) = B_{yy} \quad . \qquad \text{(I-16)}$$

Residual sum of squares (eliminating all other effects)with $n_{..}$-r-v+1 d.f.:

$$\text{Total} - SS(\mu', \rho_j', \tau_i')$$

$$= \sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{n_{ij}} Y_{ijh}^2 - \mu' Y_{...} - \sum_{j=1}^{r} \rho_j' Y_{.j.} - \sum_i \tau_i' Y_{i..} \quad . \qquad \text{(I-17)}$$

The sums of squares for a classification eliminating the effects of regression, the mean, and the other classification are given in Table 2. Alternatively, the sum of squares for the first classification adjusted for the other effects as $V_{yy}' = SS(\hat{\mu}, \hat{\tau}_i, \hat{\rho}_j, \hat{\beta}) - SS(\mu^+, \rho_j^+, \beta^+)$, where μ^+, ρ_j^+, and β^+ are obtained from the normal equations with each $\hat{\tau}_i$ set equal to zero.

The various sums of squares for the X variate are obtained in a similar manner with μ'_x, τ'_{xi}, ρ'_{xj}, μ''_x, etc. being the corresponding estimates of the effects for the X variate.

The various sums of products for the linear model given by formula (I-1) are computed as follows:

$$T_{xy} = \sum_{i=1}^{v} \sum_{j=1}^{r} \sum_{h=1}^{n_{ij}} Y_{ijh} Y_{ijh} - \frac{X_{...} Y_{...}}{n_{..}} \quad ; \qquad \text{(I-18)}$$

$$R_{xy} = \sum_{j=1}^{r} \frac{X_{.j.} Y_{.j.}}{n_{.j}} - \frac{X_{...} Y_{...}}{n_{..}} \quad ; \qquad \text{(I-19)}$$

$$A_{xy} = \sum_{i=1}^{v} \frac{X_{i..} Y_{i..}}{n_{i.}} - \frac{X_{...} Y_{...}}{n_{..}} \quad ; \qquad \text{(I-20)}$$

$$V_{xy} = \sum_{i=1}^{v} \tau'_{xi} \left\{ Y_{i\cdot\cdot} - \sum_{j=1}^{r} n_{ij}\bar{y}_{\cdot j\cdot} \right\}$$

$$= \sum_{j=1}^{v} \tau'_{i} \left\{ X_{i\cdot\cdot} - \sum_{j} n_{ij}\bar{x}_{\cdot j\cdot} \right\}$$

$$= \mu' X_{\cdot\cdot\cdot} + \Sigma\, \rho'_j X_{\cdot j\cdot} + \Sigma\, \tau'_i X_{i\cdot\cdot} - \sum_j X_{\cdot j\cdot}\bar{y}_{\cdot j\cdot}$$

$$= \mu'_x Y_{\cdot\cdot\cdot} + \Sigma\, \rho'_{x\cdot j} Y_{\cdot j\cdot} + \Sigma\, \tau'_{x\cdot i} Y_{i\cdot\cdot} - \sum_j X_{\cdot j\cdot}\bar{y}_{\cdot j} \quad ; \qquad \text{(I-21)}$$

$$D_{xy} = T_{xy} - R_{xy} - V_{xy} = T_{xy} - B_{xy} - A_{xy} \quad ; \qquad \text{(I-22)}$$

The above formulae for sums of products are applicable for the gth X variate in a multiple covariance set-up.

If the experiment is designed as a three-way classification with covariates the linear model is:[1]

$$Y_{ijhf} = \mu + \alpha_i + \gamma_j + \delta_h + \epsilon_{ijhf} + \sum_{g=1}^{b} \beta_g (X_{gijhf} - \bar{x}_g) \ , \qquad \text{(I-23)}$$

where μ = a common mean effect, α_i = effect common to the ith level of the first or A classification, γ_j = effect common to jth level of the second or C classification, δ_h = effect common to the hth level of the third or D classification, ϵ_{ijhf} = a random error effect, and β_g = partial regression of Y on X_g in the residual line in the analysis of covariance. There are no interaction terms for a Class I analysis. The normal equations for the various

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[1] This model holds for either a complete or incomplete factorial arrangement of the combinations. For an incomplete factorial arrangement (e.g. the latin square) care must be exercised in summing over the various subscripts, and the total number of observations N is not necessarily = $\sum\sum\sum_{ijh} n_{ijh} = n_{\cdot\cdot\cdot}$.

effects are given below:

$\underline{\hat{\mu}}$:

$$n_{...}\hat{\mu} + \sum_{i=1}^{a} n_{i..}\hat{\alpha}_i + \sum_{j=1}^{c} n_{.j.}\hat{\gamma}_j + \sum_{h=1}^{d} n_{..h}\hat{\delta}_h = Y_{....} \quad ; \qquad \text{(I-24)}$$

$\underline{\hat{\alpha}_i}$:

$$n_{i..}(\hat{\mu}+\hat{\alpha}_i) + \sum_j n_{ij.}\hat{\gamma}_j + \sum_h n_{i.h}\hat{\delta}_h + \sum_g \hat{\beta}_g(X_{gi...}-n_{i..}\bar{x}_g) = Y_{i...} \quad ; \qquad \text{(I-25)}$$

$\underline{\hat{\gamma}_j}$:

$$n_{.j.}(\hat{\mu}+\hat{\gamma}_j) + \sum_i n_{ij.}\hat{\alpha}_i + \sum_h n_{.jh}\hat{\delta}_h + \sum_g \hat{\beta}_g(X_{g.j..}-n_{.j.}\bar{x}_g) = Y_{.j..} \quad ; \qquad \text{(I-26)}$$

$\underline{\hat{\delta}_h}$:

$$n_{..h}(\hat{\mu}+\hat{\delta}_h) + \sum_i n_{i.h}\hat{\alpha}_i + \sum_j n_{.jh}\hat{\gamma}_j + \sum_g \hat{\beta}_g(X_{g..h.}-n_{..h}\bar{x}_g) = Y_{..h.} \quad ; \qquad \text{(I-27)}$$

$\underline{\hat{\beta}_g}$:

$$\sum_i \hat{\alpha}_i(X_{gi...}-n_{i..}\bar{x}_g) + \sum_j \hat{\gamma}_j(X_{g.j..}-n_{.j.}\bar{x}_g) + \sum_h \hat{\delta}_h(X_{g..h.}-n_{..h}\bar{x}_g)$$
$$+ \sum\sum\sum\sum_{ijhf} (X_{gijhf}-\bar{x}_g) \sum_g \hat{\beta}_g(X_{gijhf}-\bar{x}_g)$$
$$= \sum\sum\sum\sum_{ijhf=1}^{n_{ijh}} Y_{ijhf}(X_{gijhf}-\bar{x}_g) \quad . \qquad \text{(I-28)}$$

The above equations plus the following yield unique estimates of the effects:

$$\sum_{i=1}^{a} \hat{\alpha}_i = 0 \quad ; \qquad \text{(I-29)}$$

$$\sum_{j=1}^{c} \hat{\gamma}_j = 0 \quad ; \qquad \text{(I-30)}$$

$$\sum_{h=1}^{d} \hat{\delta}_h = 0 \quad . \qquad \text{(I-31)}$$

The form for the linear covariance analysis is indicated in Table 3. In general, the various sums of squares for the variance analysis are obtained as follows:

$$T_{yy} = \Sigma\Sigma\Sigma\Sigma\ Y^2_{ijhf} - SS(\mu''') \quad ; \tag{I-32}$$

$$U_{yy} = SS(\mu^*, \alpha_i^*) - SS(\mu''') \quad ; \tag{I-33}$$

$$V_{yy} = SS(\mu^-, \alpha_i^-, \gamma_j^-) - SS(\mu^*, \alpha_i^*) \quad ; \tag{I-34}$$

$$D_{yy} = SS(\mu', \alpha_i', \gamma_j', \delta_h') - SS(\mu^-, \alpha_i^-, \gamma_j^-) \quad ; \tag{I-35}$$

$$D'_{yy} = SS(\hat{\mu}, \hat{\alpha}_i, \hat{\gamma}_j, \hat{\delta}_h, \hat{\beta}_g) - SS(\mu'', \alpha_i'', \gamma_j'', \beta_g'') \quad ; \tag{I-36}$$

$$R_{yy} = T_{yy} - U_{yy} - V_{yy} - D_{yy} \quad . \tag{I-37}$$

A similar set-up is used to obtain the A_{yy} and C_{yy} sums of squares and the sums of squares for the X variate (or variates). The cross products are obtained by procedures similar to that for the two-way classification [see 9] .

The procedure is easily generalized for a q-way classification with b covariates. The algebra and the arithmetic become more difficult but the principles are the same. Detailed examples for two-way classifications are described by Das [3] and Federer [6,8] for a covariance analysis and by Yates [20], Snedecor [16], Snedecor and Cox [17] , Nair [12] , and Henderson [10] for a variance analysis. Stevens [18] gives an analysis for an unbalanced three-way classification.

In order to obtain a test of significance for two treatment means, it is necessary to compute the variances for the two means considered. Although the variances and covariances of the $Q_{i.}$ values in (I-9) are known [12], it is rather cumbersome to compute the variance of a difference for two effects,

Table 3. Analysis of covariance (linear) for a three-way classification with unequal numbers in the subclasses -- Case I

Source of variation		Sums of products		
	d.f.	y^2	xy	x^2
Total (elminiating mean)	$n_{...}-1$	T_{yy}	T_{xy}	T_{xx}
A (ignoring C and D; elim. mean)	$a-1$	U_{yy}	U_{xy}	U_{xx}
C (ign.D; elim.A and mean)	$c-1$	V_{yy}	V_{xy}	V_{xx}
D (elim.A, C, and mean)	$d-1$	D_{yy}	D_{xy}	D_{xx}
Residual (by subtraction)	f_r	R_{yy}	R_{xy}	R_{xx}
C (elim. A, D, and mean)	$c-1$	C_{yy}	C_{xy}	C_{xx}
A (elim. C, D, and mean)	$a-1$	A_{yy}	A_{xy}	A_{xx}
Residual (elim. regression, A, C, D, and mean)	f_r-1	$R'_{yy}=R_{yy}-R^2_{xy}/R_{xx}$		
Residual + D	f_r+d-2	$S'_{yy}=R_{yy}+D_{yy}-(R_{xy}+D_{xy})^2/(R_{xx}+D_{xx})$		
D (elim. regression, A, C, and mean)	$d-1$	$D'_{yy}=S'_{yy}-R'_{yy}$		
C + residual	f_r+c-2	$W'_{yy}=R_{yy}+C_{yy}-(R_{xy}+C_{xy})^2/(R_{xx}+C_{xx})$		
C (elim. regression, A, D, and mean)	$c-1$	$C'_{yy}=W'_{yy}-R'_{yy}$		
A + residual	f_r+a-2	$Z'_{yy}=A_{yy}+R_{yy}-(A_{xy}+R_{xy})/(A_{xx}+R_{xx})$		
A (elim. regression, C, D, and mean)	$a-1$	$A'_{yy}=Z'_{yy}-R'_{yy}$		

say $\hat{\tau}_1$ and $\hat{\tau}_2$, but if this is desired, Rao [15] has described the general procedure. Also, one could obtain an average coefficient for the variance of a treatment mean and then use the resulting average standard error of a mean in one of the multiple range tests [see 7, ch. II].

Case II

When interaction is present in a two-way classification with b covariates, the linear model is:

$$Y_{ijh} = \mu + \tau_i + \rho_j + \rho\tau_{ij} + \epsilon_{ijh} + \sum_{g=1}^{b} \beta_g (X_{gijh} - \bar{x}_g) \quad , \tag{II-1}$$

where μ, τ_i, ρ_j, and β_g are defined in equation (I-2) and where $\rho\tau_{ij}$ = an interaction effect common to the ijth combination of the two categories. The interaction effects are considered to be fixed effects and $n_{ij} = 1,2,\cdots$ (i.e., $n_{ij} > 0$).

The normal equations for the various effects are:

$\underline{\hat{\mu}}$:

$$n_{..}\hat{\mu} + \sum_{j=1}^{r} n_{.j}\hat{\rho}_j + \sum_{i=1}^{v} n_{i.}\hat{\tau}_i + \sum_{i=1}^{v} \sum_{j=1}^{r} n_{ij}\widehat{\rho\tau}_{ij} = Y_{...} \quad ; \tag{II-2}$$

$\underline{\hat{\tau}_i}$:

$$n_{i.}(\hat{\mu} + \hat{\tau}_i) + \sum_{j=1}^{r} n_{ij}(\hat{\rho}_j + \widehat{\rho\tau}_{ij}) + \sum_{g=1}^{b} \hat{\beta}_g (X_{gi..} - n_{i.}\bar{x}_g) = Y_{i..} \quad ; \tag{II-3}$$

$\underline{\hat{\rho}_j}$:

$$n_{.j}(\hat{\mu}+\hat{\rho}_j) + \sum_{i=1}^{v} n_{ij}(\hat{\tau}_i+\widehat{\rho\tau}_{ij}) + \sum_{g=1}^{b} \hat{\beta}_g (X_{g.j.}-n_{.j}\bar{x}_g) = Y_{.j.} \;; \tag{II-4}$$

$\underline{\widehat{\rho\tau}_{ij}}$:

$$n_{ij}(\hat{\mu}+\hat{\rho}_j+\hat{\tau}_i+\widehat{\rho\tau}_{ij}) + \sum_{g=1}^{b} \hat{\beta}_g (X_{gij.}-n_{ij}\bar{x}_g) = Y_{ij.} \quad ; \tag{II-5}$$

$\underline{\hat{\beta}_g}$:

$$\sum_i \hat{\tau}_i (X_{gi..}-n_{i.}\bar{x}_g) + \sum_j \hat{\rho}_j (X_{g.j.}-n_{.j}\bar{x}_g) + \sum\sum_{ij} \widehat{\rho\tau}_{ij} (X_{gij.}-n_{ij}\bar{x}_g)$$

$$+ \sum\sum\sum_{ijh} (X_{gijh}-\bar{x}_g) \sum_g \beta_g (X_{gijh}-\bar{x}_g) = \sum\sum\sum_{ijh} Y_{ijh}(X_{gijh}-\bar{x}_g) \quad . \tag{II-6}$$

The above l+v+r+rv+b equations plus the following result in unique solutions for the $\hat{\mu}$, $\hat{\rho}_j$, $\hat{\tau}_i$, $\widehat{\rho\tau}_{ij}$, and $\hat{\beta}_g$:

$$\sum_{i=1}^{v} \hat{\tau}_i = 0 \quad ; \tag{II-7}$$

$$\sum_{j=1}^{r} \hat{\rho}_j = 0 \quad ; \tag{II-8}$$

$$\sum_{i=1}^{v} \widehat{\rho\tau}_{ij} = 0 \quad ; \tag{II-9}$$

$$\sum_{j=1}^{r} \widehat{\rho\tau}_{ij} = 0 \quad . \tag{II-10}$$

If b = 1, then an estimate of $\beta = \beta_1$ is obtained from the within sub-classes line in the analysis of covariance (Table 4) as:

$$\hat{\beta} = \frac{\sum_{i=1}^{v} \sum_{j=1}^{r} \left\{ \sum_{h=1}^{n_{ij}} X_{ijh} Y_{ijh} - X_{ij.} Y_{ij.}/n_{ij} \right\} = S_{xy}}{\sum_{i=1}^{v} \sum_{j=1}^{r} \left\{ \sum_{h=1}^{n_{ij}} X_{ijh}^2 - X_{ij.}^2/n_{ij} \right\} = S_{xx}} \tag{II-11}$$

If b > 1 then the various $\hat{\beta}_g$ may be obtained from the following b equations [see 16, sec. 13.7]:

$$\hat{\beta}_1 \sum_{ijh}\sum\sum (X_{gijh}-\bar{x}_{gij.})(X_{1ijh}-\bar{x}_{1ij.}) + \hat{\beta}_2 \sum_{ijh}\sum\sum (X_{gijh}-\bar{x}_{gij.})(X_{2ijh}-\bar{x}_{2ij.})$$

$$+ \cdots + \hat{\beta}_b \sum_{ijh}\sum\sum (X_{gijh}-\bar{x}_{gij.})(X_{bijh}-\bar{x}_{bij.}) = \sum_{ijh}\sum\sum (Y_{ijh}-\bar{y}_{ij.})(X_{gijh}-\bar{x}_{gij.}) \tag{II-12}$$

for g = 1,2,$\cdots$,b.

With the above estimates for $\hat{\beta}_g$, the remaining effects are estimated as follows:

$$\hat{\mu} = \frac{1}{rv} \sum_{i=1}^{v} \sum_{j=1}^{r} (\bar{y}_{ij.} - \sum_g \hat{\beta}_g (\bar{x}_{gij.} - \bar{x}_g)) \; ; \tag{II-13}$$

$$\hat{u} + \hat{\tau}_i = \frac{1}{r} \sum_{j=1}^{r} (\bar{y}_{ij.} - \sum_g \hat{\beta}_g (\bar{x}_{gij.} - \bar{x}_g)); \tag{II-14}$$

$$\hat{\mu} + \hat{\rho}_j = \frac{1}{v} \sum_{i=1}^{v} (\bar{y}_{ij.} - \sum_g \hat{\beta}_g (\bar{x}_{gij.} - \bar{x}_g)) \; ; \tag{II-15}$$

$$\widehat{\rho\tau}_{ij} = \bar{y}_{ij.} - \hat{\mu} - \hat{\rho}_j - \hat{\tau}_i - \sum_g \hat{\beta}_g (\bar{x}_{gij.} - \bar{x}_g) \quad . \tag{II-16}$$

If one or more of the $n_{ij} = 0$, no analysis is possible unless a zero (or some other constant value) is inserted for the corresponding $\widehat{\rho\tau}_{ij}$; this results in the following changes for formulae (II-9) and (II-10) [8,10] :

$$\sum_{i=1}^{v_j} \widehat{\rho\tau}_{ij} = 0 \quad ; \tag{II-17}$$

$$\sum_{j=1}^{r_i} \widehat{\rho\tau}_{ij} = 0 \quad ; \tag{II-18}$$

The summation is from $j = 1,2,\cdots,r_i$ in (II-3) and from $i = 1,2,\cdots,v_j$ in (II-4). These changes result in a biased analysis. The size of the bias depends upon the actual values of $\rho\tau_{ij}$ which are estimated as zero and upon the number of n_{ij} equal to zero in the analysis. In some situations, the only recourse is to use the biased Case II analysis, to use a Case I analysis assuming no interaction, or to use an among subclasses and within subclasses analysis.

Before computing the various sums of products it should be noted that there may be no interest in discussing the category means when interaction is assumed present. If there is interest in the category means, then weighted means may be more appropriate than the unweighted means. If an among and a within groups variance or covariance analysis is suitable (i.e., the rv or $r_. = \sum_i r_i = \sum_j v_j = v.$ subclass means are to be compared), then there is little difficulty in computing the among and within subclasses analysis and in comparing the subclass means, $\bar{y}_{ij.}$, either by an F test or a multiple range test [see 7, ch. II] /1.

If tests of significance are required for category means or if components of variance are to be estimated by Henderson's Method 3 [11], the various sums of products in Table 4 need to be computed. The first three rows of sums of products in Table 4 are obtained from a Case I analysis (i.e., forumlae (I-12), (I-13), (I-15), (I-18), (I-19), and (I-21)). The within subclasses sums of products are:

$$S_{yy} = \sum_{ij}\sum \left\{ \sum_h Y^2_{ijh} - Y^2_{ij.}/n_{ij} \right\} ; \qquad \text{(II-19)}$$

$$S_{xy} = \sum_{ij}\sum \left\{ \sum_h Y_{ijh}X_{ijh} - X_{ij.}Y_{ij.}/n_{ij} \right\} ; \qquad \text{(II-20)}$$

$$S_{xx} = \sum_{ij}\sum \left\{ \sum_h X^2_{ijh} - X^2_{ij.}/n_{ij} \right\} . \qquad \text{(II-21)}$$

/1 In a multiple range test, the standard error of a mean used to compute a significant range for comparing two means, say $\bar{y}_{1..}$ and $\bar{y}_{2..}$ with $n_{1.}$ and $n_{2.}$ observations, respectively, is (D. B. Duncan, written correspondence):

$$\sqrt{\frac{S_{yy}}{n_{..}-r_.}\left\{\frac{1}{2}\left(\frac{1}{n_{1.}} + \frac{1}{n_{2.}}\right)\right\}}$$

If the two means are compared in a covariance analysis, the factor $S'_{yy}/(n_{..}-r_.-1)$ replaces $S_{yy}/(n_{..}-r_.)$. Also, one could use $1/\sqrt{n_{1.}n_{2.}}$ as the approximation instead of $\frac{1}{2}(\frac{1}{n_{1.}} + \frac{1}{n_{2.}})$.

The interaction sums of products are computed as:

$$I_{yy} = SS(\bar{\mu},\bar{\rho}_j,\bar{\tau}_i,\overline{\rho\tau}_{ij}) - SS(\mu',\rho_j',\tau_i')$$

$$= \sum_{ij}\sum \frac{Y_{ij\cdot}^2}{n_{ij}} - \mu' Y_{\cdots} - \Sigma\rho_j' Y_{\cdot j\cdot} - \Sigma\tau_i' Y_{i\cdot\cdot} \quad , \qquad \text{(II-22)}$$

where the estimates μ', ρ_j', and τ_i' are defined in the paragraph preceding formula (I-12) and where $\bar{\mu}$, $\bar{\rho}_j$, $\bar{\tau}_i$, $\overline{\rho\tau}_{ij}$ are the estimates obtained from formulae (II-1) to (II-10) when each $X_{gijh} - \bar{x}_g$ is set equal to zero,

$$I_{xy} = \sum_{ij}\sum \frac{X_{gij\cdot}Y_{ij\cdot}}{n_{ij\cdot}} - \mu' X_{g\cdots} - \Sigma\rho_j' X_{g\cdot j\cdot} - \Sigma\tau_i' X_{gi\cdot\cdot}$$

$$= \sum_{ij}\sum \frac{X_{gij\cdot}Y_{ij\cdot}}{n_{ij}} - \mu_x' Y_{\cdots} - \Sigma\rho_{xj}' Y_{\cdot j\cdot} - \Sigma\tau_{xi}' Y_{i\cdot\cdot} \quad ; \qquad \text{(II-23)}$$

$$I_{xx} = \sum_{ij}\sum \frac{X_{gij\cdot}^2}{n_{ij}} - \mu_x' X_{g\cdots} - \Sigma\rho_{xj}' X_{g\cdot j\cdot} - \Sigma\tau_{xi}' X_{gi\cdot\cdot} \quad . \qquad \text{(II-24)}$$

The sums of products for the first factor (eliminating mean, second factor and interaction) is [10,21]:

$$A_{yy} = SS(\bar{\mu}, \bar{\rho}_j, \bar{\tau}_i, \overline{\rho\tau}_{ij}) - SS(\mu^+, \rho_j^+, \rho\tau_{ij}^+)$$

$$= \Sigma w_{i\cdot}\bar{\bar{y}}_{i\cdot\cdot}^2 - \frac{(\Sigma w_{i\cdot}\bar{\bar{y}}_{i\cdot\cdot})^2}{\Sigma w_{i\cdot}} \quad , \qquad \text{(II-25)}$$

where μ^+, ρ_j^+, $\rho\tau_{ij}^+$ are the estimates obtained by minimizing the following sum of squares:

$$\sum_{ijh}\sum\sum(Y_{ijh} - \mu - \rho_j - \rho\tau_{ij})^2 + \lambda_1\Sigma\rho_j$$

$$+ \sum_i \lambda_{2i} \sum_j \rho\tau_{ij} + \sum_j \lambda_{3j} \sum_i \rho\tau_{ij} \quad ; \qquad \text{(II-26)}$$

λ_1, λ_{2i}, and λ_{3j} are Lagrangian multipliers [12]. Also, use could be made of

Table 4. Analysis of covariance (linear) for a two-way classification with unequal numbers in the subclasses -- Case II (first factor = A; second factor = B).

Source of variation	d.f.	Sums of products y^2	xy	x^2
Total (eliminating mean)	$n_{..}-1$	T_{yy}	T_{xy}	T_{xx}
B (elim. mean; ign. A and AxB	r-1	R_{yy}	R_{xy}	R_{xx}
A (elim. B and mean; ign. AxB	v-1	V_{yy}	V_{xy}	V_{xx}
AxB (elim. A, B, and mean)	(r-1)(v-1)	I_{yy}	I_{xy}	I_{xx}
Within subclasses = S	$n_{..}-rv$	S_{yy}	S_{xy}	S_{xx}
B (elim. A, AxB, and mean)	r-1	B_{yy}	B_{xy}	B_{xx}
A (elim. B, AxB, and mean)	v-1	A_{yy}	A_{xy}	A_{xx}
S (adj. for regression)	$n_{..}-rv-1$	$S'_{yy} = S_{yy} - S^2_{xy}/S_{xx}$		
S + B	$n_{..}-rv+r-2$	$U'_{yy} = B_{yy} + S_{yy} - (B_{xy} + S_{xy})^2/(B_{xx} + S_{xx})$		
B (adj. for other effects)	r-1	$B'_{yy} = U'_{yy} - S'_{yy}$		
S + A	$n_{..}-rv+v-2$	$W'_{yy} = A_{yy} + S_{yy} - (A_{xy} + B_{xy})^2/(A_{xx} + B_{xx})$		
A (adj. for other effects)	v-1	$A'_{yy} = W'_{yy} - S'_{yy}$		
S + A x B	$n_{..}-r-v$	$Z'_{yy} = I_{yy} + S_{yy} - (I_{xy} + S_{xy})^2/(I_{xx} + S_{xx})$		
A x B (adj. for other effects)	(r-1)(v-1)	$I'_{yy} = Z'_{yy} - S'_{yy}$		

equations (II-8), (II-9), and (II-10) in the following sum of squares prior to differentiation:

$$\sum_{ijh}\Sigma\Sigma(Y_{ijh} - \mu - \rho_j - \rho\tau_{ij})^2 \tag{II-27}$$

In this way there would be the same number of independent equations as there are unknowns. The number of equations to be solved would be $1 + r - 1 + (r-1)(v-1) = rv - v + 1$.

Also,

$$A_{xy} = \sum_i w_{i.}\bar{\bar{x}}_{i..}\bar{\bar{y}}_{i..} - \frac{\Sigma w_{i.}\bar{\bar{x}}_{i..}\Sigma w_{i.}\bar{\bar{y}}_{i..}}{\Sigma w_{i.}} \;; \tag{II-28}$$

$$A_{xx} = \sum_i w_{i.}\bar{\bar{x}}^2_{i..} - \frac{(\Sigma w_{i.}\bar{\bar{x}}_{i..})^2}{\Sigma w_{i.}} \;. \tag{II-29}$$

In the above, $\bar{\bar{y}}_{i..} = \frac{1}{r}\sum_{j=1}^{r}\bar{y}_{ij.}$, $\bar{\bar{x}}_{i..} = \frac{1}{r}\sum_{j=1}^{r}\bar{x}_{ij.}$, and $w_{i.} = (\frac{1}{r^2}\sum_j \frac{1}{n_{ij}})^{-1}$.

Due a theorem given by Yates [22], the F or z test of significance is appropriate for comparing the two mean squares $A_{yy}/(v-1)$ and $S_{yy}/(n_{..}-rv)$. Likewise, $F = \frac{A'_{yy}(n_{..}-rv-1)}{(v-1)S'_{yy}}$ is a valid test of the null hypothesis of zero effects in a covariance analysis.

Similarly,

$$B_{yy} = \sum_j w_{.j}\bar{\bar{y}}_{.j.} - \frac{(\sum_j w_{.j}\bar{\bar{y}}_{.j.})^2}{\sum_j w_{.j}} \;; \tag{II-30}$$

$$B_{xy} = \sum_j w_{.j}\bar{\bar{y}}_{.j.}\bar{\bar{x}}_{.j.} - \frac{\sum_j w_{.j}\bar{\bar{x}}_{.j.}\sum_j w_{.j}\bar{\bar{y}}_{.j.}}{\Sigma w_{.j}} \;; \tag{II-31}$$

$$B_{xx} = \sum_j w_{.j}\bar{\bar{x}}_{.j.}^2 - \frac{\sum_j(w_{.j}\bar{\bar{x}}_{.j.})^2}{\Sigma w_{.j}} \;; \tag{II-32}$$

where $\bar{\bar{y}}_{.j.} = \frac{1}{v}\sum_{i=1}^{v}\bar{y}_{ij.}$, $\bar{\bar{x}}_{.j.} = \frac{1}{v}\sum_{i=1}^{v}\bar{x}_{ij.}$, and $w_{.j} = (\frac{1}{v^2}\sum_{i=1}^{v}\frac{1}{n_{ij}})^{-1}$.

A test of the null hypothesis for the levels of the second classification may be made by comparing $B_{yy}/(r-1)$ with $S_{yy}/(n_{..}-rv)$ in the variance analysis or $B'_{yy}/(r-1)$ with $S'_{yy}/(n_{..}-rv-1)$ in the covariance analysis.

For a three-way classification with b covariates and interaction effects not assumed to be zero as in (I-23), the linear model is:

$$Y_{ijhf} = \mu + \alpha_i + \gamma_j + \delta_h + \alpha\gamma_{ij} + \alpha\delta_{ih} + \gamma\delta_{jh} + \alpha\gamma\delta_{ijh} + \epsilon_{ijhf} + \sum_{g=1}^{b} \beta_g (X_{gijhf} - \bar{x}_g) \quad , \tag{II-33}$$

where μ, α_i, γ_j, δ_h, ϵ_{ijhf}, and β_g are defined in (I-23) and where the remaining terms represent interaction effects (fixed) between the three classifications. Here, n_{ijh} = number of observations for the ijhth combination of the three factors must be greater than zero. If any $n_{ijh} = 0$, then the alternatives described for the preceding unbalanced two-way classification may be used.

The normal equations for the various effects are:

$\hat{\mu}$:

$$n_{...}\hat{\mu} + \sum_{i=1}^{a} n_{i..}\hat{\alpha}_i + \sum_{j=1}^{c} n_{.j.}\hat{\gamma}_j + \sum_{h=1}^{d} n_{..h}\hat{\delta}_h + \sum_{i=1}^{a}\sum_{j=1}^{c} n_{ij.}\widehat{\alpha\gamma}_{ij}$$

$$+ \sum_{i=1}^{a}\sum_{h=1}^{d} n_{i.h}\widehat{\alpha\delta}_{ih} + \sum_{j=1}^{c}\sum_{h=1}^{d} n_{.jh}\widehat{\gamma\delta}_{jh} + \sum_{i=1}^{a}\sum_{j=1}^{c}\sum_{h=1}^{d} n_{ijh}\widehat{\alpha\gamma\delta}_{ijh}$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{c}\sum_{h=1}^{d}\sum_{f=1}^{n_{ijh}} Y_{ijhf} = Y_{....} \quad ; \tag{II-34}$$

$\hat{\alpha}_i$:

$$n_{i..}(\hat{\mu} + \hat{\alpha}_i) + \sum_j n_{ij.}(\hat{\gamma}_j + \widehat{\alpha\gamma}_{ij}) + \sum_h n_{i.h}(\hat{\delta}_h + \widehat{\alpha\delta}_{ih}) + \sum_{jh}\sum n_{ijh}(\widehat{\gamma\delta}_{jh} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g (X_{gi...} - n_{i..}\bar{x}_g) = Y_{i...} \quad ; \tag{II-35}$$

$\underline{\hat{\gamma}_j}$:

$$n_{\cdot j\cdot}(\hat{\mu} + \hat{\gamma}_j) + \sum_i n_{ij\cdot}(\hat{\alpha}_i + \widehat{\alpha\gamma}_{ij}) + \sum_h n_{\cdot jh}(\hat{\delta}_h + \widehat{\gamma\delta}_{jh}) + \sum\sum_{ih} n_{ijh}(\widehat{\alpha\delta}_{ih} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g (X_{g\cdot j\cdot\cdot} - n_{\cdot j\cdot}\bar{x}_g) = Y_{\cdot j\cdot\cdot} \quad ; \qquad \text{(II-36)}$$

$\underline{\hat{\delta}_h}$:

$$n_{\cdot\cdot h}(\hat{\mu} + \hat{\delta}_h) + \sum_i n_{i\cdot h}(\hat{\alpha}_i + \widehat{\alpha\delta}_{ih}) + \sum_j n_{\cdot jh}(\hat{\gamma}_j + \widehat{\gamma\delta}_{jh}) + \sum\sum_{ij} n_{ijh}(\widehat{\alpha\gamma}_{ij} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g(X_{g\cdot\cdot h\cdot} - n_{\cdot\cdot h}\bar{x}_g) = Y_{\cdot\cdot h\cdot} \quad ; \qquad \text{(II-37)}$$

$\underline{\widehat{\alpha\gamma}_{ij}}$:

$$n_{ij\cdot}(\hat{\mu} + \hat{\alpha}_i + \hat{\gamma}_j + \widehat{\alpha\gamma}_{ij}) + \sum_h n_{ijh}(\hat{\delta}_h + \widehat{\alpha\delta}_{ih} + \widehat{\gamma\delta}_{jh} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g(X_{gij\cdot\cdot} - n_{ij\cdot}\bar{x}_g) = Y_{ij\cdot\cdot} \quad ; \qquad \text{(II-38)}$$

$\underline{\widehat{\alpha\delta}_{ih}}$:

$$n_{i\cdot h}(\hat{\mu} + \hat{\alpha}_i + \hat{\delta}_h + \widehat{\alpha\delta}_{ih}) + \sum_j n_{ijh}(\hat{\gamma}_j + \widehat{\alpha\gamma}_{ij} + \widehat{\gamma\delta}_{jh} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g(X_{gi\cdot h\cdot} - n_{i\cdot h}\bar{x}_g) = Y_{i\cdot h\cdot} \quad ; \qquad \text{(II-39)}$$

$\underline{\widehat{\gamma\delta}_{jh}}$:

$$n_{\cdot jh}(\hat{\mu} + \hat{\gamma}_j + \hat{\delta}_h + \widehat{\gamma\delta}_{jh}) + \sum_i n_{ijh}(\hat{\alpha}_i + \widehat{\alpha\gamma}_{ij} + \widehat{\alpha\delta}_{ih} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g(X_{g\cdot jh\cdot} - n_{\cdot jh}\bar{x}_g) = Y_{\cdot jh\cdot} \quad ; \qquad \text{(II-40)}$$

$\widehat{\alpha\gamma\delta}_{ijh}$:

$$n_{ijh}(\hat{\mu} + \hat{\alpha}_i + \hat{\gamma}_j + \hat{\delta}_h + \widehat{\alpha\gamma}_{ij} + \widehat{\alpha\delta}_{ih} + \widehat{\gamma\delta}_{jh} + \widehat{\alpha\gamma\delta}_{ijh})$$

$$+ \sum_g \hat{\beta}_g(X_{gijh\cdot\cdot} - n_{ijh}\bar{x}_g) = Y_{ijh\cdot} \quad ; \qquad \text{(II-41)}$$

$\hat{\beta}_g$:

$$\sum_{i=1}^{a} \hat{\alpha}_i(X_{gi\cdots} - n_{i\cdot\cdot}\bar{x}_g) + \sum_{j=1}^{c} \hat{\gamma}_j(X_{g\cdot j\cdot\cdot} - n_{\cdot j\cdot}\bar{x}_g) + \sum_{h=1}^{d} \hat{\delta}_h(X_{g\cdot\cdot h\cdot} - n_{\cdot\cdot h}\bar{x}_g)$$

$$+ \sum_{ij}\sum \widehat{\alpha\gamma}_{ij}(X_{gij\cdot\cdot} - n_{ij\cdot}\bar{x}_g) + \sum_{ih}\sum \widehat{\alpha\delta}_{ih}(X_{gi\cdot h\cdot} - n_{i\cdot h}\bar{x}_g) + \sum_{jh}\sum \widehat{\gamma\delta}_{jh}(X_{g\cdot jh\cdot} - n_{\cdot jh}\bar{x}_g)$$

$$+ \sum_{ijh}\sum\sum \widehat{\alpha\gamma\delta}_{ijh}(X_{gijh\cdot} - n_{ijh}\bar{x}_g) + \sum_{ijhf}\sum\sum\sum (X_{gijhf} - \bar{x}_g) \sum_g \hat{\beta}_g(X_{gijhf} - \bar{x}_g)$$

$$= \sum_{i=1}^{a} \sum_{j=1}^{c} \sum_{h=1}^{d} \sum_{f=1}^{n_{ijh}} Y_{ijhf}(X_{gijhf} - \bar{x}_g) \quad . \qquad \text{(II-42)}$$

The following restrictions are placed on the above normal equations:

$$\sum_{i=1}^{a} \hat{\alpha}_i = 0 \quad ; \qquad \text{(II-43)}$$

$$\sum_{j=1}^{c} \hat{\gamma}_j = 0 \quad ; \qquad \text{(II-44)}$$

$$\sum_{h=1}^{d} \hat{\delta}_h = 0 \quad ; \qquad \text{(II-45)}$$

$$\sum_{i=1}^{a} \widehat{\alpha\gamma}_{ij} = \sum_{j=1}^{c} \widehat{\alpha\gamma}_{ij} = 0 \quad ; \qquad \text{(II-46)}$$

$$\sum_{i=1}^{a} \widehat{\alpha\delta}_{ih} = \sum_{h=1}^{d} \widehat{\alpha\delta}_{ih} = 0 \quad ; \qquad \text{(II-47)}$$

$$\sum_{j=1}^{c} \widehat{\gamma\delta}_{jh} = \sum_{h=1}^{d} \widehat{\gamma\delta}_{jh} = 0 \quad ; \qquad \text{(II-48)}$$

$$\sum_{i=1}^{a} \widehat{\alpha\gamma\delta}_{ijh} = \sum_{j=1}^{c} \widehat{\alpha\gamma\delta}_{ijh} = \sum_{h=1}^{d} \widehat{\alpha\gamma\delta}_{ijh} = 0 \quad . \qquad \text{(II-49)}$$

Solution of the above equations yields unique estimates for the 1 + (a-1) + (c-1) + (d-1) + (a-1)(c-1) + (a-1)(d-1) + (c-1)(d-1) + (a-1)(c-1)(d-1) + b effects. The sum of squares due to the effects in a variance analysis when each $(X_{gijhf} - \bar{x}_g)$ is set equal to zero is the sum of squares of subclass totals; thus,

$$SS(\mu', \alpha_i', \gamma_j', \delta_h', \alpha\gamma_{ij}', \alpha\delta_{ih}', \gamma\delta_{jh}', \alpha\gamma\delta_{ijh}') = \sum_{ijh}\sum\sum Y^2_{ijh\cdot}/n_{ijh} \qquad \text{(II-50)}$$

with acd degrees of freedom; this sum of squares minus $Y^2_{\cdots\cdot}/n_{\cdots}$ is the among subclasses sum of squares.

The sum of squares due to one effect, say the second factor main effect, eliminating all other effects in a variance analysis is:

$$\sum_{ijh}\sum\sum Y^2_{ijh\cdot}/n_{ijh} - SS(\mu^-, \alpha_i^-, \delta_h^-, \alpha\gamma_{ij}^-, \alpha\delta_{ih}^-, \gamma\delta_{jh}^-, \alpha\gamma\delta_{ijh}^-) = \sum_{ijh}\sum\sum Y^2_{ijh\cdot}/n_{ij\cdot}$$

$$- \Big\{ \mu^- Y_{\cdots\cdot} + \sum \alpha_i^- Y_{i\cdots} + \sum \delta_h^- Y_{\cdot\cdot h} + \sum_{ij}\sum \alpha\gamma_{ij}^- Y_{ij\cdot\cdot} + \sum_{ih}\sum \alpha\delta_{ih}^- Y_{i\cdot h\cdot}$$

$$+ \sum_{jh}\sum \gamma\delta_{jh}^- Y_{\cdot jh\cdot} + \sum_{ijh}\sum\sum \alpha\gamma\delta_{ijh}^- Y_{ijh\cdot} \Big\} \quad , \qquad \text{(II-51)}$$

where the $^-$ estimates are obtained from equations (II-34) to (II-49) setting each $\hat{\gamma}_j$ and each $(X_{gijhf} - \bar{x}_g)$ equal to zero and solving the remaining equations (omitting (II-36), (II-42) and (II-44)). The above sum of squares may be obtained simply as:

$$\sum w_{\cdot j\cdot} \bar{\bar{y}}^2_{\cdot j\cdot\cdot} - \frac{(\sum w_{\cdot j\cdot} \bar{\bar{y}}_{\cdot j\cdot\cdot})^2}{\sum w_{\cdot j\cdot}} \qquad \text{(II-52)}$$

where $\bar{\bar{y}}_{\cdot j\cdot\cdot} = \frac{1}{ad} \sum_{ih}\sum \bar{y}_{ijh\cdot}$ and where $w_{\cdot j\cdot} = \left(\frac{1}{a^2d^2} \sum_{ih}\sum \frac{1}{n_{ijh}}\right)^{-1}$. Similar sums of squares may be used for the other two main effect sums of squares. Also, the sum of squares due to A, C, and AxC eliminating all other effects is:

$$\sum_{ijh}\sum\sum \frac{Y^2_{ijh\cdot}}{n_{ijh}} - SS(\mu^+,\ \delta^+_h,\ \alpha\delta^+_{ih},\ \gamma\delta^+_{jh},\ \alpha\gamma\delta^+_{ijh})\ , \qquad \text{(II-53)}$$

where the $^+$ effects are estimated from the remaining normal equations with each $\hat{\alpha}_i$, $\hat{\gamma}_j$, $\widehat{\alpha\gamma}_{ij}$, and $(X_{gijhf} - \bar{\bar{x}}_g)$ set equal to zero. By analogy, this sum of squares should be of the form:

$$\sum\sum w_{ij\cdot}\bar{\bar{y}}^2_{ij\cdot} - \frac{(\sum\sum w_{ij\cdot}\bar{\bar{y}}_{ij\cdot})^2}{\sum\sum w_{ij}} - \left(\sum_i w_{i\cdot\cdot}\bar{\bar{y}}^2_{i\cdot\cdot} - \frac{(\sum w_{i\cdot\cdot}\bar{\bar{y}}_{i\cdot\cdot})^2}{\sum w_{i\cdot\cdot}}\right)$$

$$- \left(\sum_j w_{\cdot j\cdot}\bar{\bar{y}}^2_{\cdot j\cdot} - \frac{(\sum w_{\cdot j\cdot}\bar{\bar{y}}_{\cdot j\cdot})^2}{\sum w_{\cdot j\cdot}}\right) - \text{correction for disproportion}, \qquad \text{(II-54)}$$

where the items in the equation are defined in a manner similar to equation (II-52). The sums of squares for the various effects except the mean are presented in Table 5. The method of computation for sums of squares which add to the total is given in the upper part of the table. The resulting mean squares from the next to the last line in the upper part and all those from the lower part of Table 5 are used in making tests of significance of the null hypotheses for each of the main effects and interactions. The within sub-classes mean square is the error mean square for all F tests.

As stated previously, an among groups and within groups analysis with a multiple range test might suffice in certain instances. The procedure would be similar to that described for the two-way classification.

Covariance analyses are carried out in the same manner as for the two-way classification. For example, the sum of cross products for the AxC (eliminating all other effects) is:

Table 5. Analysis of variance for an unbalanced three-way classification -- Case II (first factor = A; second factor = C; third factor = D).

Source of variation	Degrees of freedom	Sum of squares /1
Total (elim. mean only)	$n_{..}-1$	$\Sigma\Sigma\Sigma\Sigma Y^2_{ijhf}-Y^2_{....}/n_{...}=T-CT$
A(elim. mean ign. other effects)	a-1	$\Sigma Y^2_{i...}/n_{i..}-CT$
C(elim. mean and A; ign. others)	c-1	$SS(\mu^{(1)},\alpha_i^{(1)},\gamma_j^{(1)})-\Sigma Y^2_{i...}/n_{i..}$
AxC(elim.A,C,mean; ign. others)	(a-1)(c-1)	$\sum\limits_{ij}\Sigma Y^2_{ij..}/n_{ij.}-SS(\mu^{(1)},\alpha_i^{(1)},\gamma_j^{(1)})$
D(elim.A,C,AxC,mean; ign. others)	d-1	$SS(\mu^{(2)},\alpha_i^{(2)},\gamma_j^{(2)},\alpha\gamma_{ij}^{(2)},\delta_h^{(2)})-\sum\limits_{ij}\Sigma Y^2_{ij..}/n_{ij..}=II-I$
AxD(elim.A,C,AxC,D, mean; ign. CxD, AxCxD)	(a-1)(d-1)	$SS(\mu^{(3)},\alpha_i^{(3)},\gamma_j^{(3)},\alpha\gamma_{ij}^{(3)},\delta_h^{(3)},\alpha\delta_{ih}^{(3)})-II=III-II$
CxD(elim. all effects but AxCxD)	(c-1)(d-1)	$SS(\mu^{(4)},\alpha_i^{(4)},\gamma_j^{(4)},\alpha\gamma_{ij}^{(4)},\delta_h^{(4)},\alpha\delta_{ih}^{(4)},\gamma\delta_{jh}^{(4)})-III=IV-III$
AxCxD(elim. all other effects)	(a-1)(c-1)(d-1)	$\sum\limits_{ijh}\Sigma\Sigma Y^2_{ijh.}/n_{ijh}-IV$
Within subclasses	$n_{...}-acd$	$\sum\limits_{ijh}\Sigma\Sigma\left\{\sum\limits_f Y^2_{ijhf}-Y^2_{ijh.}/n_{ij}\right\}$
A(elim. all other effects)	a-1	similar to formula (II-52)
C(elim. all other effects)	c-1	formula (II-52)
AxC(elim. all other effects)	(a-1)(c-1)	$\Sigma\Sigma\Sigma Y^2_{ijh.}/n_{ijh}-SS(\mu^{(5)},\alpha_i^{(5)},\gamma_j^{(5)},\delta_h^{(5)},\alpha\delta_{ih}^{(5)},\gamma\delta_{jh}^{(5)},\alpha\gamma\delta_{ijh}^{(5)})$
D(elim. all other effects)	d-1	similar to formula (II-52)
AxD(elim. all other effects)	(a-1)(d-1)	similar to AxC (elim. all other effects)
CxD(elim. all other effects)	(c-1)(d-1)	similar to AxC (elim. all other effects)

/1 The superscript in parentheses refers to the estimates obtained from equations (II-34) to (II-49) setting the effect(s) not included in the term SS( ) equal to zero and omitting the equations for these effects from equations (II-34) to (II-49).

$$\sum_{ijh}\sum\sum \frac{Y_{ijh\cdot}X_{ijh\cdot}}{n_{ijh}} - \Big\{ \mu^{(5)} X_{\cdots\cdot} + \Sigma\, \alpha_i^{(5)} X_{i\cdots} + \sum_j \gamma_j^{(5)} X_{\cdot j\cdot\cdot}$$

$$+ \Sigma\, \delta_h^{(5)} X_{\cdot\cdot h\cdot} + \sum_{ih}\sum \alpha\delta_{ih}^{(5)} X_{i\cdot h\cdot} + \sum_{jh}\sum \gamma\delta_{jh}^{(5)} X_{\cdot jh\cdot}$$

$$+ \sum_{ijh}\sum\sum \alpha\gamma\delta_{ijh}^{(5)} X_{ijh\cdot} \Big\}, \qquad \text{(II-55)}$$

where the estimates with the superscript $^{(5)}$ are obtained from equations (II-34) to (II-49) omitting equations (II-38), (II-42), and (II-46) with each $\widehat{\alpha\gamma}_{ij}$ and each $(X_{gijhf} - \bar{x}_g)$ set equal to zero.

With b covariates, solutions for $\hat{\beta}_g$ may be obtained from formulae similar to (II-12), i.e., the within subclasses sums of squares and cross products for a three (or higher) - way classification. Likewise, a direct extension of the results in Tables 4 and 5 results in a variance or covariance analysis for a q-way classification.

Case III

If one or both categories in a two-way classification are considered to be a sample of levels or treatments from a large population of levels or treatments for the given category, the interaction mean square is used to test hypotheses about mean effects and to adjust the means for variation due to a covariate(s). Also, the interaction effects, as such, may not be of much interest except in estimating the component of variance associated with interaction effects. Since a weighted effect would be preferable, for statistical reasons, to an unweighted effect, the problem of estimating effects is complicated because the weights are unknown. If the weights are estimated from the data, further statistical problems arise. In certain experiments, the experimenter may have little or no control over the number of observations in each subclass and it may be realistic to assume that the linear model is of the following form:

$$Y_{ijh} = \mu + \tau_i + \rho_j + \rho\tau_{ij} + \epsilon_{ijh} + \beta_1(\bar{x}_{ij.} - \bar{\bar{x}}) + \beta(X_{ijh} - \bar{x}_{ij.}), \quad \text{(III-1)}$$

where μ, τ_i, ρ_j, and ϵ_{ijh}, $\bar{\bar{x}}$, and X_{ijh} are defined in (II-1) except that τ_i and/or ρ_j are considered to be random effects, $\bar{x}_{ij.} = \sum_{h=1}^{n_{ij}} X_{ijh}/n_{ij}$, $\rho\tau_{ij}$ = an interaction effect of the ijth combination of the two categories and is a random effect, β_1 = regression coefficient from the interaction line in the covariance analysis, and β = a within subclasses regression coefficient; if $\beta_1 = \beta$ then (III-1) reduces to (II-1) except for the random effects set-up for (III-1) as opposed to the fixed effects assumption in (II-1). For such situations, a Case III analysis may be desired and the weights may have to be estimated from data in the present or in a past experiment. The weights themselves will have a sampling distribution but the experiment would usually be analyzed assuming that the estimated weights were the true weights.

Two sums of squares could be minimized to obtain least squares estimates of the effects. These are:

$$\sum_{i=1}^{v}\sum_{j=1}^{r}\sum_{h=1}^{n_{ij}}(Y_{ijh} - \mu - \tau_i - \rho_j - \rho\tau_{ij} - \beta_1(\bar{x}_{ij.} - \bar{x}) - \beta(X_{ijh} - \bar{x}_{ij.}))^2 \qquad \text{(III-2)}$$

and

$$\sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}(\bar{y}_{ij.} - \mu - \tau_i - \rho_j - \beta_1(\bar{x}_{ij.} - \bar{x}))^2 \quad , \qquad \text{(III-3)}$$

where $\bar{y}_{ij.}$ and $\bar{x}_{ij.}$ = subclass means for the two variates. If $w_{ij} = n_{ij}$ in (III-3) where n_{ij} = number of observations in the ijth subclass, minimization of (III-3) results in the same estimates of μ, τ_i, ρ_j, and β_1 obtained from minimizing (III-2). Since this is true, since w_{ij} is not always equal to n_{ij}, and since β is estimated from the within subclasses sums of products, formula (II-11), the sum of squares in (III-3) is minimized instead of the one in (III-2).

The problem of correct weighting is relatively simple if the true weights are known. Thus, weighting inversely to the variance of $\bar{y}_{ij.}$ results in:

$$w_{ij} = \frac{1}{\sigma^2_{\rho\tau} + \sigma^2_{\epsilon}/n_{ij}} = \frac{n_{ij}}{n_{ij}\sigma^2_{\rho\tau} + \sigma^2_{\epsilon}} \quad , \qquad \text{(III-4)}$$

where $\sigma^2_{\rho\tau}$ and σ^2_{ϵ} are the variance components associated with interaction and within subclasses, respectively. In practice σ^2_{ϵ} and $\sigma^2_{\rho\tau}$ are usually unknown. First, consider the two limiting situations:

(i) $\sigma^2_{\rho\tau}$ is large relative to $\sigma^2_{\epsilon}/n_{ij}$;

(ii) $\sigma^2_{\rho\tau}$ is small relative to $\sigma^2_{\epsilon}/n_{ij}$.

In situation (i), one may set $w_{ij} = 1$ for all practical purposes; i.e., an analysis of covariance is performed on the subclass means. In the second situation, w_{ij} is set equal to n_{ij}, and the analysis goes through as described below using the weights w_{ij} [also, see 8].

The true situation is usually in between the above two limiting situations, and in order to perform an analysis on the data it will be necessary to have reasonably good estimates of $\sigma^2_{\rho\tau}$ and σ^2_{ϵ}. In many situations the estimated variance (or covariance) components will need to be estimated from the data themselves. One method for doing this is illustrated by Federer [8], where Henderson's [11] Method 1 was used. This method assumes that both effects are random effects; if one of the effects is fixed then a bias results [11]. Recourse to Henderson's [11] methods 2 or 3 may be made if one of the effects is considered to be fixed.

The normal equations resulting from a minimization of (III-3) are:

$\underline{\hat{\mu}}$:

$$w_{..}\hat{\mu} + \sum_{i=1}^{v} w_{i.}\hat{\tau}_i + \sum_{j=1}^{r} w_{.j}\hat{\rho}_j + \hat{\beta}_1 \sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}(\bar{x}_{ij.}-\bar{x}) = \sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}\bar{y}_{ij.} \; ; \qquad \text{(III-5)}$$

$\underline{\hat{\tau}_i}$:

$$w_{i.}(\hat{\mu} + \hat{\tau}_i) + \sum_{j=1}^{r} w_{ij}(\hat{\rho}_j + \hat{\beta}_1(\bar{x}_{ij.}-\bar{x})) = \sum_{j=1}^{r} w_{ij}\bar{y}_{ij.} \; ; \qquad \text{(III-6)}$$

$\underline{\hat{\rho}_j}$:

$$w_{.j}(\hat{\mu} + \hat{\rho}_j) + \sum_{i=1}^{v} w_{ij}(\hat{\tau}_i + \hat{\beta}_1(\bar{x}_{ij.}-\bar{x})) = \sum_{i=1}^{v} w_{ij}\bar{y}_{ij.} \; ; \qquad \text{(III-7)}$$

$\underline{\hat{\beta}_1}$:

$$\hat{\mu}\sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}(\bar{x}_{ij.}-\bar{x}) + \sum_{i=1}^{v}\hat{\tau}_i \sum_{j=1}^{r} w_{ij}(\bar{x}_{ij.}-\bar{x}) + \sum_{j=1}^{r}\hat{\rho}_j \sum_{i=1}^{v} w_{ij}(\bar{x}_{ij.}-\bar{x})$$

$$+ \hat{\beta}_1 \sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}(\bar{x}_{ij.}-\bar{x})^2 = \sum_{i=1}^{v}\sum_{j=1}^{r} w_{ij}\bar{y}_{ij.}(\bar{x}_{ij.}-\bar{x}) \; . \qquad \text{(III-8)}$$

Applying the restrictions in (II-7) and (II-8), results in the following:

$$\hat{\mu} + \hat{\rho}_j = \frac{1}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}(\bar{y}_{ij\cdot} - \hat{\beta}_1(\bar{x}_{ij\cdot} - \bar{x}) - \hat{\tau}_i) \quad ; \tag{III-9}$$

$$\hat{\mu} = \frac{1}{r} \sum_j \frac{1}{w_{\cdot j}} \sum_i w_{ij}(\bar{y}_{ij\cdot} - \hat{\beta}_1(\bar{x}_{ij\cdot} - \bar{x}) - \hat{\tau}_i) \quad ; \tag{III-10}$$

$$\hat{\mu} + \hat{\tau}_i = \frac{1}{w_{i\cdot}} \sum_{j=1}^{r} w_{ij}(\bar{y}_{ij\cdot} - \hat{\beta}_1(\bar{x}_{ij\cdot} - \bar{x}) - \hat{\rho}_j) \quad ; \tag{III-11}$$

$$\hat{\beta}_1 = \frac{\sum\sum w_{ij}\bar{x}_{ij\cdot}\bar{y}_{ij\cdot} - \sum_j \frac{1}{w_{\cdot j}}(\sum_i w_{ij}\bar{y}_{ij})(\sum_i w_{ij}\bar{x}_{ij\cdot}) - \sum_i \hat{\tau}_i \sum_j w_{ij}\bar{x}_{ij\cdot} + \sum_j \frac{1}{w_{\cdot j}} \sum_i w_{ij}\hat{\tau}_i \sum_i w_{ij}\bar{x}_{ij\cdot}}{\sum\sum_{ij} w_{ij}\bar{x}_{ij\cdot}^{\,2} - \sum_j (\sum_i w_{ij}\bar{x}_{ij\cdot})^2 / w_{\cdot j}} \; . \tag{III-12}$$

In the above $w_{\cdot\cdot} = \sum_{i=1}^{v} \sum_{j=1}^{r} w_{ij}$; $w_{i\cdot} = \sum_{j=1}^{r} w_{ij}$; $w_{\cdot j} = \sum_{i=1}^{v} w_{ij}$; w_{ij} = weight for ijth subclass mean; w_{ij} could be zero and the analysis could still be performed. That is, zero observations could be obtained for certain subclasses. If this was a random event, the interaction effect for that subclass would be set equal to its expected value = zero.

Substituting for $\hat{\mu}$, $\hat{\rho}_j$, and $\hat{\beta}_1$ results in v equations in the $\hat{\tau}_i$, the kth equation being:

$$\hat{\tau}_k \left\{ w_{k\cdot} - \sum_{j=1}^{r} w_{kj}^{\,2}/w_{\cdot j} - \frac{1}{G_{xx}} \left(\sum_{j=1}^{r} w_{kj}\bar{x}_{kj\cdot} - \sum_{j=1}^{r} \frac{w_{kj}}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij\cdot} \right)^2 \right\}$$

$$- \sum_{\substack{i=1 \\ \neq k}}^{v} \hat{\tau}_i \left\{ \sum_{j=1}^{r} \frac{w_{kj}w_{ij}}{w_{\cdot j}} - \frac{1}{G_{xx}} \left(\sum_{j=1}^{r} w_{kj}\bar{x}_{kj\cdot} - \sum_{j=1}^{r} \frac{w_{kj}}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij\cdot} \right) \right.$$

$$\left. \left(\sum_{j=1}^{r} \frac{w_{ij}}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij} - \sum_{j=1}^{r} w_{ij}\bar{x}_{ij\cdot} \right) \right\} = \sum_{j=1}^{r} w_{kj}\bar{y}_{kj\cdot} - \sum_{j=1}^{r} \frac{w_{kj}}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}\bar{y}_{ij\cdot} - \frac{G_{xy}}{G_{xx}} \left(\sum_{j=1}^{r} w_{kj}\bar{x}_{kj\cdot} \right.$$

$$\left. - \sum_{j=1}^{r} \frac{w_{kj}}{w_{\cdot j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij\cdot} \right) \; , \tag{III-13}$$

where

$$G_{xy} = \sum_{i=1}^{v} \sum_{j=1}^{r} w_{ij}\bar{x}_{ij.}\bar{y}_{ij.} - \sum_{j=1}^{r} \frac{1}{w_{.j}}\left(\sum_{i=1}^{v} w_{ij}\bar{y}_{ij.}\right)\left(\sum_{i=1}^{v} w_{ij}\bar{x}_{ij.}\right) \qquad \text{(III-14)}$$

and

$$G_{xx} = \sum_{i=1}^{v} \sum_{j=1}^{r} w_{ij}\bar{x}_{ij.}^{\,2} - \sum_{j=1}^{r} \left(\sum_{i=1}^{v} w_{ij}\bar{x}_{ij.}\right)^2 / w_{.j} \quad . \qquad \text{(III-15)}$$

Likewise, the v equations in the $\hat{\tau}_i$ in terms of $\hat{\beta}_1$ and the observations are:

$$\begin{pmatrix} w_{1.} - \Sigma w_{1j}^2/w_{.j} & -\Sigma w_{1j}w_{2j}/w_{.j} & -\Sigma w_{1j}w_{3j}/w_{.j} & \cdots & -\Sigma w_{1j}w_{vj}/w_{.j} \\ -\Sigma w_{1j}w_{2j}/w_{.j} & w_{2.} - \Sigma w_{2j}^2/w_{.j} & -\Sigma w_{2j}w_{3j}/w_{.j} & \cdots & -\Sigma w_{2j}w_{vj}/w_{.j} \\ \vdots & & & \ddots & \vdots \\ -\Sigma w_{1j}w_{vj}/w_{.j} & -\Sigma w_{2j}w_{vj}/w_{.j} & -\Sigma w_{3j}w_{vj}/w_{.j} & \cdots & w_{v.} - \Sigma w_{vj}^2/w_{.j} \end{pmatrix} \begin{pmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_v \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{r} w_{1j}\bar{y}_{1j.} - \sum_{j=1}^{r} \frac{w_{1j}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{y}_{ij.} - \hat{\beta}_1\left(\sum_{j=1}^{r} w_{1j}\bar{x}_{1j.} - \sum_{j=1}^{r} \frac{w_{1j}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij.}\right) \\ \sum_{j=1}^{r} w_{2j}\bar{y}_{2j.} - \sum_{j=1}^{r} \frac{w_{2j}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{y}_{ij.} - \hat{\beta}_1\left(\sum_{j=1}^{r} w_{2j}\bar{x}_{2j.} - \sum_{j=1}^{r} \frac{w_{2j}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij.}\right) \\ \vdots \\ \sum_{j=1}^{r} w_{vj}\bar{y}_{vj.} - \sum_{j=1}^{r} \frac{w_{vj}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{y}_{ij.} - \hat{\beta}_1\left(\sum_{j=1}^{r} w_{vj}\bar{x}_{vj.} - \sum_{j=1}^{r} \frac{w_{vj}}{w_{.j}} \sum_{i=1}^{v} w_{ij}\bar{x}_{ij.}\right) \end{pmatrix} \quad . \qquad \text{(III-16)}$$

The v equations in (III-16) plus equation (II-7) result in a solution for the $\hat{\tau}_i$. If a variance analysis is desired each $\bar{x}_{ij.}$ in (III-16) is set equal to zero and the resulting estimates τ_i' of the τ_i are obtained. $\hat{\beta}_1$ in the covariance analysis is the interaction sum of cross products divided by the interaction sum of squares for the covariate, i.e. $\hat{\beta}_1 = E_{xy}/E_{xx}$ (Table 6).

The sums of products in the first three rows of Table 6 are obtained in the usual manner (see formulae (I-12),(I-18),(II-19),(II-20), and (II-21). The formulae for the remaining sums of the products are given below.

Among subclasses with r.-1 degrees of freedom:

$$W_{yy} = \Sigma\Sigma w_{ij}\bar{y}^2_{ij.} - \frac{(\Sigma\Sigma w_{ij}\bar{y}_{ij.})^2}{w_{..}} \quad ; \qquad \text{(III-17)}$$

(similarly for W_{xx})

$$W_{xy} = \Sigma\Sigma w_{ij}\bar{y}_{ij.}\bar{x}_{ij.} - \frac{(\Sigma\Sigma w_{ij}\bar{x}_{ij.})(\Sigma\Sigma w_{ij}\bar{y}_{ij.})}{w_{..}} \quad ; \qquad \text{(III-18)}$$

where $r_. = \sum_{i=1}^{v} r_i = v_. = \sum_{j=1}^{r} v_j$ = number of subclasses with one or more observations per subclass.

B (eliminating mean; ignoring A) with r-1 d.f.:

$$R_{yy} = \sum_{j=1}^{r} \frac{(\sum_i w_{ij}\bar{y}_{ij.})^2}{w_{.j}} - \frac{(\Sigma\Sigma w_{ij}\bar{y}_{ij.})^2}{w_{..}} \quad ; \qquad \text{(III-19)}$$

(similarly for R_{xx})

$$R_{xy} = \sum_{j=1}^{r} \frac{(\sum_i w_{ij}\bar{y}_{ij.})(\sum_i w_{ij}\bar{x}_{ij.})}{w_{.j}} - \frac{(\Sigma\Sigma w_{ij}\bar{x}_{ij.})(\sum\sum_{ij} w_{ij}\bar{y}_{ij.})}{w_{..}} \quad . \qquad \text{(III-20)}$$

A (eliminating mean and B) with v-1 d.f.:

$$A_{yy} = SS(\mu', \rho'_j, \tau'_i) - SS(\mu'', \rho''_j) = \mu'\Sigma\Sigma w_{ij}\bar{y}_{ij.} + \sum_j \rho'_j(\sum_i w_{ij}\bar{y}_{ij.}) + \sum_i \tau'_i(\sum_j w_{ij}\bar{y}_{ij.})$$

$$- \sum_j(\sum_i w_{ij}\bar{y}_{ij.})^2/w_{.j} \qquad \text{(similarly for } A_{xx}) \quad ; \qquad \text{(III-21)}$$

$$A_{xy} = \mu'\sum\sum_{ij} w_{ij}\bar{x}_{ij.} + \sum_j \rho'_j(\sum_i w_{ij}\bar{x}_{ij.}) + \sum_i \tau'_i(\sum_j w_{ij}\bar{x}_{ij.}) - \sum_j(\sum_i w_{ij}\bar{y}_{ij.})(\sum_i w_{ij}\bar{x}_{ij.})/w_{.j}$$

$$= \mu'_x \sum\sum_{ij} w_{ij}\bar{y}_{ij.} + \sum_j \rho'_{xj}(\sum_i w_{ij}\bar{y}_{ij.}) + \sum_i \tau'_{xi}(\sum_j w_{ij}\bar{y}_{ij.}) - \sum_j(\sum_i w_{ij}\bar{y}_{ij.})(\sum_i w_{ij}\bar{x}_{ij.})/w_{.j} \quad ,$$

(III-22)

Table 6. Covariance (linear) analysis for an unbalanced two-way classification -- Case III.

Source of variation	d.f.	Sums of products y^2	xy	x^2
Total (elim. mean only)	$n_{..}-1$	T_{yy}	T_{xy}	T_{xx}
Within subclasses	$n_{..}-r_{.}$	S_{yy}	S_{xy}	S_{xx}
Among subclasses	$r_{.}-1$	by subtraction		
Among weighted subclass totals	$r_{.}-1$	W_{yy}	W_{xy}	W_{xx}
B (elim. mean; ign. A)	$r-1$	R_{yy}	R_{xy}	R_{xx}
A (elim. mean and B) = A	$v-1$	A_{yy}	A_{xy}	A_{xx}
Error (interaction) = E	f_e	E_{yy}	E_{xy}	E_{xx}
B (elim. mean and A) = B	$r-1$	B_{yy}	B_{xy}	B_{xx}
E (elim. A,B, mean,and regression)	f_e-1	$E'_{yy}=E_{yy}-E^2_{xy}/E_{xx}$		
A + E	f_e+v-2	$U'_{yy}=A_{yy}+E_{yy}-(A_{xy}+E_{xy})^2/(A_{xx}+E_{xx})$		
A (elim. B,mean, and regression)	$v-1$	$A'_{yy}=U'_{yy}-E'_{yy}$		
B + E	f_e+r-2	$V'_{yy}=B_{yy}+E_{yy}-(B_{xy}+E_{xy})^2/(B_{xx}+E_{xx})$		
B (elim. A,mean, and regression)	$r-1$	$B'_{yy}=V'_{yy}-E'_{yy}$		

where μ', ρ'_j, and τ'_i are the estimates obtained from equations (III-5),(III-6), and (III-7) when each $(\bar{x}_{ij.}-\bar{x})$ is set equal to zero and where μ'' and ρ''_j are the estimates obtained from these equations (except (III-6)) when each $\hat{\tau}_i$ and each $(\bar{x}_{ij.}-\bar{x})$ are set equal to zero; the μ'_x, ρ'_{xj}, and τ'_{xi} are similar estimates obtained for the X variate.

Error (interaction eliminating A, B, and mean) with $f_e=r_.-r-v+1$ d.f.:

$$E_{yy} = \Sigma\Sigma w_{ij}\bar{y}^2_{ij.} - SS(\mu',\rho'_j,\tau'_i) = W_{yy} - R_{yy} - A_{yy} \quad ; \qquad \text{(III-23)}$$

$$E_{xy} = W_{xy} - R_{xy} - A_{xy} \quad ; \qquad \text{(III-24)}$$

$$E_{xx} = W_{xx} - R_{xx} - A_{xx} \quad . \qquad \text{(III-25)}$$

B (eliminating A and mean) with r-1 d.f.:

$$B_{yy} = SS(\mu',\rho'_j,\tau'_i) - SS(\mu''',\tau'''_i) = SS(\mu',\rho'_j,\tau'_i) - \sum_i(\sum_j w_{ij}\bar{y}_{ij.})^2/w_{i.} \quad ; \qquad \text{(III-26)}$$

(similarly for B_{xx})

$$B_{xy} = \mu'\Sigma\Sigma w_{ij}\bar{x}_{ij.} + \sum_j\rho'_j(\sum_i w_{ij}\bar{x}_{ij.}) + \sum_i\tau'_i(\sum_j w_{ij}\bar{x}_{ij.})$$

$$- \sum_i(\sum_j w_{ij}\bar{y}_{ij.})(\sum_i w_{ij}\bar{x}_{ij.})/w_{i.} \quad , \qquad \text{(III-27)}$$

where μ''' and τ'''_i are obtained from formula (III-5) and (III-7) with each $\hat{\rho}_j$ and each $(\bar{x}_{ij.}-\bar{x})$ set equal to zero. The procedure for obtaining the sums of squares eliminating regression is indicated in Table 6.

The procedure for b covariates is a straightforward extension of the results for a Case II analysis. The normal equations given by formulae (III-5) to (III-8) would be altered for the terms involving $\hat{\beta}_1$ and the $\bar{x}_{ij.}$'s. Instead of the term $\hat{\beta}_1(\bar{x}_{ij.}-\bar{x})$ substitute $\sum_{g=1}^{b}\hat{\beta}_{1g}(\bar{x}_{gij.}-\bar{x}_g)$ in (III-1) and make the corresponding changes in formulae (III-5) to (III-16). The $\hat{\beta}_{1g}$ would be obtained from the Error (interaction) line in the multiple covariance analysis as described for a Case II analysis.

For a three-way classification and for the random effects situation for all three categories, additional statistical problems arise in testing hypotheses about main effects. For example, consider that the linear model is:

$$Y_{ijhf}=\mu+\alpha_i+\gamma_j+\delta_h+\alpha\gamma_{ij}+\alpha\delta_{ih}+\gamma\delta_{jh}+\alpha\gamma\delta_{ijh}+\epsilon_{ijhf}+\beta_1(\bar{x}_{ij..}-\bar{x}_{i...}-\bar{x}_{.j..}+\bar{x})$$
$$+\beta_2(\bar{x}_{i.h.}-\bar{x}_{i...}-\bar{x}_{..h.}+\bar{x})+\beta_3(\bar{x}_{.jh.}-\bar{x}_{.j..}-\bar{x}_{..h.}+\bar{x})+\beta_4(\bar{x}_{ijh.}+2\bar{x}_{i...}+2\bar{x}_{.j..}+2\bar{x}_{..h.}$$
$$-\bar{x}_{ij..}-\bar{x}_{i.h.}-\bar{x}_{.jh.}-4\bar{x})+\beta(X_{ijhf}-\bar{x}_{ijh.}) \quad , \tag{III-28}$$

where the effects are defined in (II-33) except that all effects are random effects, β_1,β_2,β_3 and β_4 are different regression coefficients from the various interaction lines in the analysis of covariance. In such a model, there is no single line in the analysis of covariance that is suitable for testing hypotheses about main effects and for adjusting for variation in the covariate. A simplifying assumption would be to assume $\beta_1=\beta_2=\beta_3=\beta_4$ and possibly that $\beta_4=\beta$ also. This would simplify the covariance problem but would not simplify the hypotheses testing problem [7, ch. VIII]. A simplification in hypothesis testing would be possible if one or more of the two-factor interactions could be assumed relatively small, and then there would be a single interaction mean square which could be used to test hypotheses about main effects concerned.

The normal equations and the Case III covariance analysis for the unbalanced three-way classification represent a straightforward extension of the Case II analysis for a three-way classification and of a Case III analysis for a two-way classification. Solutions for the various effects are obtained as before.

Estimation of variance components from a given set of data, either the present or a past experiment, presents some problems even for an unbalanced two-way classification. As a first approximation in an experiment one could use Henderson's [11] method 1 assuming both effects random and assuming w_{ij} fixed. The coefficients for the various parameters for different sums of squares are listed in Table 7. When $w_{ij} = n_{ij}$, some of the coefficients reduce to a simpler form. The process is repeated on each analysis and the new weights are used in the subsequent analysis. The process is repeated until the weights stabilize.

Alternatively, one could use Henderson's [11] method 3 and obtain the expectation of I'_{yy} in Table 4 as a first estimate for $\hat{\sigma}^2_{\rho\tau}$. σ^2_ϵ is estimated from the within subclasses mean squares as $\hat{\sigma}^2_\epsilon = S'_{yy}/(n_{..}-rv-1)$. The analysis of covariance in Table 6 is obtained. As a second estimate of $\sigma^2_{\rho\tau}$ one could obtain the expected value of E'_{yy} in Table 6 for fixed w_{ij}. Then, Table 6 could be recomputed using the second estimate of $\sigma^2_{\rho\tau}$ and $\hat{\sigma}^2_\epsilon$. Table 6 could be recomputed and a third estimate of $\sigma^2_{\rho\tau}$ obtained. The process could be repeated until estimates of $\sigma^2_{\rho\tau}$ stabilize.

As a third procedure, Federer [8] suggested that the expectation of $I'_{yy}/(r_{.}-r-v+1)$ in Table 4 be set equal to:

$$\sigma^2_\epsilon + k_0\sigma^2_{\rho\tau} , \tag{III-29}$$

where

$$k_0 = \frac{1}{r_{.}-r-v+1}\left\{ n_{..}+\sum_{ij}\sum n^2_{ij}\left(\frac{1}{n_{..}} - \frac{1}{n_{i.}} - \frac{1}{n_{.j}}\right) - \frac{I_{xx}}{I_{xx}+S_{xx}} \right\} ; \tag{III-30}$$

also $\hat{\sigma}^2_{\rho\tau}$ could be estimated from E'_{yy} in Table 6 as:

$$\hat{\sigma}^2_{\rho\tau} = \frac{1}{k_0}\left[\frac{E'_{yy}}{\Sigma\Sigma w_{ij}/n_{ij}+\sum_{ij}\sum w^2_{ij}/n_{ij}(1/w_{..}-1/w_{i.}-1/w_{.j})} - \hat{\sigma}^2_\epsilon \right] . \tag{III-31}$$

Table 7. Conditional expectations of sums of squares for a two-way classification with unequal numbers in the subclasses -- Case III, w_{ij} fixed.

Sum of squares	Parameter and coefficient					
	μ^2	σ_τ^2	σ_ρ^2	σ_ϵ^2	σ_δ^2	β_1^2
(1) $\sum\sum_{ij} w_{ij}\bar{y}_{ij.}^2$	$w_{..}$	$w_{..}$	$w_{..}$	$w_{..}$	$\sum\sum_{ij} \frac{w_{ij}}{n_{ij}}$	$\sum\sum_{ij} w_{ij}(\bar{x}_{ij.}-\bar{x})^2$
(2) $\sum_i(\sum_j w_{ij}\bar{y}_{ij.})^2/w_{i.}$	$w_{..}$	$w_{..}$	$\sum_i \frac{1}{w_{i.}} \sum_j w_{ij}^2$	$\sum_i \frac{1}{w_{i.}} \sum_j w_{ij}^2$	$\sum_i \frac{1}{w_{i.}} \sum_j \frac{w_{ij}^2}{n_{ij}}$	$\sum_i(\sum_j w_{ij}(\bar{x}_{ij.}-\bar{x}))^2/w_{i.}$
(3) $\sum_j(\sum_i w_{ij}\bar{y}_{ij.})^2/w_{.j}$	$w_{..}$	$\sum_j \frac{1}{w_{.j}} \sum_i w_{ij}^2$	$w_{..}$	$\sum_j \frac{1}{w_{.j}} \sum_i w_{ij}^2$	$\sum_j \frac{1}{w_{.j}} \sum_i \frac{w_{ij}^2}{n_{ij}}$	$\sum_j(\sum_i w_{ij}(\bar{x}_{ij.}-\bar{x}))^2/w_{.j}$
(4) $(\sum\sum_{ij} w_{ij}\bar{y}_{ij.})^2/w_{..}$	$w_{..}$	$\frac{1}{w_{..}} \sum_i w_{i.}^2$	$\frac{1}{w_{..}} \sum_j w_{.j}^2$	$\frac{1}{w_{..}} \sum\sum_{ij} w_{ij}^2$	$\frac{1}{w_{..}} \sum\sum_{ij} \frac{w_{ij}^2}{n_{ij}}$	0
(5) $(\sum\sum_{ij} w_{ij}\bar{y}_{ij.}(\bar{x}_{ij.}-\bar{x}))^2$	0	$\sum_i(\sum_j w_{ij}(\bar{x}_{ij.}-\bar{x}))^2$	$\sum_j(\sum_i w_{ij}(\bar{x}_{ij.}-\bar{x}))^2$	$\sum\sum_{ij} w_{ij}^2(\bar{x}_{ij.}-\bar{x})^2$	$\sum\sum_{ij}\frac{w_{ij}^2}{n_{ij}}(\bar{x}_{ij.}-\bar{x})^2$	$(\sum\sum_{ij} w_{ij}(\bar{x}_{ij.}-\bar{x})^2)^2$

The analogous result for an equal numbers analysis [see 7, ch. XVI] led to the coefficient k_0. Both the preceding method and Henderson's [11] method 1 are illustrated with a numerical example by Federer [8].

For a three-way or higher-way classification Henderson's [11] method 1 probably should be used to obtain the estimated variance components if the degrees of freedom are fairly large, say greater than 20 to 30, for each sum of squares considered. Also, the three-way classification could be collapsed into a two-way classification for certain experimental situations in order to simplify the analysis. In still other situations, a Case II analysis might be necessary for two factors whereas the third factor and interactions with the third factor would involve a Case III analysis.

Discussion

Variance and covariance analyses for unbalanced classifications contain computational and statistical difficulties. In order to simplify these difficulties it is suggested that the experimenter use an among groups and a within groups analysis wherever possible. If this is not a realistic analysis then the only recourse is to use the analysis for the situation involved; i.e. use a Case I analysis, Case II analysis, Case III analysis, or some combination of these three analyses. Statistical theory is lacking for some parts of a Case III analysis, but it is felt that the approximate analysis obtained by estimating the weights from the experiment itself will be suitable for analyses with greater than 20 to 30, say, degrees of freedom in the interaction or error sum of squares.

Throughout the manuscript restrictions of the form $\Sigma\hat{\tau}_i = 0 = \Sigma\hat{\rho}_j$, etc., were imposed. It might be simpler computationally to replace the restriction $\Sigma\hat{\tau}_i = 0$ by $\hat{\tau}_v = 0$ (or any other $\hat{\tau}_i$), and the restriction $\Sigma\hat{\rho}_j = 0$ by $\hat{\rho}_r = 0$. In a Case I analysis, say, the differences among the $\hat{\tau}_i$ and among the $\hat{\rho}_j$ will remain the same as before, but individual $\hat{\tau}_i$ will be less by the value of $\hat{\tau}_v$ and the $\hat{\rho}_j$ values will be less by $\hat{\rho}_r$. The estimated $\hat{\mu}$ will now be increased by the sum $\hat{\rho}_r + \hat{\tau}_v$. Since differences among the $\hat{\tau}_i$ and $\hat{\rho}_j$ rather than the estimates themselves are usually the items of interest, the criterion for selection between the restrictions $\Sigma\hat{\tau}_i = 0 = \Sigma\hat{\rho}_j$ and $\hat{\tau}_v = 0 = \hat{\rho}_r$ would be ease of computation. Stevens [18] makes use of these restrictions in a variance analysis of an unbalanced tri-factorial experiment.

Also, restrictions of the form $\Sigma n_{i.}\hat{\tau}_i = 0$, $n_{i.} \neq$ a constant, could be used *if* the experimenter were interested *only* in differences among the $\hat{\tau}_i$. The use of this restriction in an unbalanced q-way classification biases the estimates of the differences among the remaining effects, but the differences among the $\hat{\tau}_i$ are the same as when the restrictions $\Sigma\hat{\tau}_i = 0$ or $\hat{\tau}_v = 0$ are used.

Use of restrictions of the form $\widehat{\rho\tau}_{ir} = 0 = \widehat{\rho\tau}_{vj}$ for every i and every j in a two-way classification leads to different estimates of effects than do the restrictions $\sum_i \widehat{\rho\tau}_{ij} = 0 = \sum_j \rho\tau_{ij}$. In fact, the use of the restrictions $\widehat{\rho\tau}_{ir} = \widehat{\rho\tau}_{vj} = \hat{\tau}_v = \hat{\rho}_r = 0$ involves setting $\hat{\mu}$ equal to $\bar{y}_{vr.}$ in a two-way classification. The set of restrictions used should not bias the estimates of the effects. Otherwise, any set of restrictions may be used.

Numerical Examples[1]

An experiment was conducted on the effect of different humidity treatments planted at different times (6/15, 7/15, 8/15, and 9/15) on roses over a period of months. One characteristic measured was the number of saleable roses per month. The data for Y = number of saleable roses in Tables 8 and 10 represent a selected sample of data from the total experiment. The X covariate represents the position on the greenhouse bench and is used to control variation within a replicate [see 7, section XVI-11]. The two replicates were on opposite greenhouse benches. For the data in Table 8, the positions were grouped into 5 sets of 3 each and given the numbers 1 to 5. A Case I analysis of covariance for these data and the treatment means adjusted for regression are presented in Table 9. Since this is a 2 x 5 table, Snedecor's [16, section 11.11] analysis of variance procedure may be followed for the Y and X variates, and for the cross products, or the entire analysis follows from a direct application of the formulae given for a Case I analysis.

The average standard error of a ~~difference between two~~ treatment mean~~s~~ adjusted for regression may be taken as

$$s'_{\bar{y}} = \sqrt{\frac{E'_{yy}}{n_{..}-r-v}\left\{\frac{1}{n_{i.}} + \frac{V_{xx}}{n_{i.}(v-1)D_{xx}}\right\}}$$

$$= \sqrt{\frac{72.26}{3}\left\{1 + \frac{22.889}{(5-1)(9.333)}\right\}} = 6.233 \quad ,$$

and Tukey's hsd is [7, section II-1.1.4]

$$q_{05}s'_{\bar{y}} = 4.89(6.233) = 30.48 \doteq \text{hsd}.$$

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[1] The data for this example were obtained through the courtesy of Dr. R. W. Langhans, Dept. of Floriculture, Cornell University, from an experiment reported in his doctoral dissertation.

Table 8. Number of saleable flowers (roses), Y_{ijh}, opened in one month (December) and location on greenhouse bench, X_{ijh} -- 6/15 planting.

	Rep. I					Rep. II					Total		
Treatment	n_{i1}	Y	X	Y	X	n_{i2}	Y	X	Y	X	$n_{i.}$	$Y_{i..}$	$X_{i..}$
1	2	27	5	18	5	1	19	4	-	-	3	64	14
2	1	31	3	-	-	2	52	5	57	5	3	140	13
3	1	34	2	-	-	2	52	1	33	1	3	119	4
4	1	38	4	-	-	2	60	2	45	2	3	143	8
5	1	31	1	-	-	2	50	3	40	3	3	121	7
Total	6	$Y_{.1.}=179; X_{.1.}=20$				9	$Y_{.2.}=408; X_{.2.}=26$				15	587	46

Table 9. Case I covariance analysis for data in Table 8.

Source of variation	d.f.	Sums of products y^2	xy	x^2
Total (uncorrected)	15	25407	1755	174
Correction for mean	1	22971.27	1800.133	141.067
Total	14	2435.73	- 45.133	32.933
Replicate(ign.treat.;elim.mean)	1	864.90	- 24.800	0.711
Treatment(elim.mean and rep.)	4	912.70	- 47.667	22.889
Residual	9	658.13	27.334	9.333

	d.f.	adj. sum of squares	mean square
Residual(elim.reg.,treat.,rep. and mean)	8	578.08	72.26
Treat. + residual	12	1558.00	--
Treatment(elim.reg.,rep.,and mean)	4	979.92	244.98

$\hat{\mu} + \hat{\tau}_1 = 18.55$ $\quad\hat{\mu} + \hat{\tau}_3 = 42.85$ $\quad\hat{\mu} + \hat{\tau}_5 = 40.58$

$\hat{\mu} + \hat{\tau}_2 = 41.06$ $\quad\hat{\mu} + \hat{\tau}_4 = 46.94$ $\quad\hat{\rho}_1 = -5.7 = -\hat{\rho}_2$

If any two means differ by more than 30.48 = hsd, the two means are said to come from populations with different means. (The error rate is approximately five percent per experiment.) Likewise, we could compare treatment 1, the standard, with the mean of the remaining 4, $\frac{41.06+42.85+46.94+40.58}{4} = 42.86$, as follows:

$$4.89\sqrt{\frac{72.26}{2}\left(\frac{1}{3}+\frac{1}{12}+\frac{(14/3-32/12)^2}{4.333}\right)} = 27.02$$

$$\text{hsd} \doteq 4.89\sqrt{\frac{72.26}{2}\left(1+\frac{(14/3-32/12)^2}{(9.333)}\right)\left[\left(\frac{1}{3}+\frac{1}{12}\right)\right]} = \cancel{20.91}\ 27.02.$$

Since the mean of the four treatments, 42.86, minus the mean of the standard, 18.55, ~~exceeds~~ is close to the hsd it is stated that for the 6/15 planting date the ~~mean of the~~ treatments as a group ~~produced~~ significantly more saleable roses than did the standard for the month of December.

The data in Table 10 are the number of saleable roses obtained for the month of April from the 7/15 planting, Y, and the position on the greenhouse bench, X. The 15 positions were not grouped as they were in Table 8, but represent the location of the treatment in one of the 15 possible positions. It is assumed, as before, that there was a linear gradient from one end of the bench to the other. The Case II covariance analysis for the data of Table 10 is presented in Table 11, and is obtained by a direct application of the formulae given for a Case II analysis. The adjusted treatment means are 80.78, 74.15, 81.02, 63.78, and 61.78 for treatments 1,2,3,4, and 5 respectively. The approximate hsd for a comparison of treatments 1,2, and 3 with treatments 4 and 5 (a comparison suggested by the nature of the treatments) is:

$$6.29\sqrt{\frac{57.38}{2}\left(\frac{1}{9}+\frac{1}{6}+\frac{(77/9-37/6)^2}{4.00}\right)} =$$

$$\text{hsd} \doteq 6.29\sqrt{\frac{57.38}{2}\left(1+\frac{(77/9-37/6)^2}{(4.00)}\right)\left[\frac{1}{9}+\frac{1}{6}\right]} = \cancel{23.24}\ 43.99,$$

where 6.29 is the q_{05} for 5 treatments and 4 degrees of freedom in error.

Table 10. Number of saleable flowers (roses), Y_{ijh}, opened in one month (April) and location on greenhouse bench, X_{ijh} -- 7/15 planting.

	Rep. I					Rep. II					Total		
Treatment	n_{i1}	Y	X	Y	X	n_{i2}	Y	X	Y	X	n_{i3}	$Y_{i..}$	$X_{i..}$
1	1	102	15	-	-	2	71	10	79	11	3	252	36
2	2	84	9	81	7	1	76	14	-	-	3	241	30
3	2	67	5	83	4	1	74	2	-	-	3	224	11
4	1	71	11	-	-	2	51	4	63	5	3	185	20
5	1	53	2	-	-	2	63	8	61	7	3	177	17
Total	7	$Y_{.1.}=541;X_{.1.}=53$				8	$Y_{.2.}=538;X_{.2.}=61$				15	1079	114

Table 11. Case II analysis of data in Table 10.

Source of vatiation	d.f.	Sums of products y^2	xy	x^2
Total(elim. mean)	14	2426.93	447.60	229.60
Replicate(ign.tr.and inter.; elim. mean)	1	376.00	- 2.01	0.01
Treatment(elim.rep. and mean; ign.inter.)	4	1320.97	303.81	136.12
I=Interaction(elim.tr.,rep.,and mean)	4	491.46	139.80	89.47
S=Within subclasses	5	238.50	6.00	4.00
T=Treatment(elim.rep.,inter., and mean)	4	1606.67	367.33	172.53

	d.f.	adj. sum of squares	mean square
S(adjusted for regression)	4	229.50	57.38
S+I(" " ")	8	502.53	--
I(" " ")	4	273.03	68.26
S+T(" " ")	8	1055.64	--
T(" " ")	4	826.14	206.54

The difference between the means of the two groups of treatments is less than the approximate hsd = ~~2.24~~, and hence declared to be a non-significant difference.

43.99

The data from Table 10 are used to illustrate a Case III linear covariance analysis. The computational form for the analysis is given in Table 12. The first set of weights are obtained from the variance components estimated from the data in Table 10. The estimated variance components are (formulae (III-29) to (III-31)):

$$\hat{\sigma}^2_{\epsilon} = 57.38 \; ;$$

$$\hat{\sigma}^2_{\rho\tau} = \frac{1}{1.014}\left\{68.26 - 57.38\right\} = 10.73 \; ,$$

where

$$k_o = \frac{1}{4}\left(15 + \frac{25}{15} - \frac{25}{3} - \frac{11}{7} - \frac{14}{8} - \frac{89.47}{89.47 + 4.00}\right) = 1.014 \; .$$

Since n_{ij} = 1 or 2 the two weights are (formula (III-4))

$$w_{11} = \frac{1}{10.73 + 57.38} = .015$$

and

$$w_{12} = \frac{2}{2(10.73) + 57.38} = .025 \quad .$$

Since there are only two weights, the coded weights $w_{11} = 3$ and $w_{12} = 5$ will be used to simplify the computations. The actual weights may be simpler to use than coded weights in examples where the n_{ij} vary considerably.

With the computed weights Table 12 can now be completed. Applying the formulae given for a Case III analysis, Table 13 is constructed. The various estimates used to obtain the sums of products are the μ', τ'_i, ρ'_j, μ'_x, τ'_{xi}, and ρ'_{xj} vlaues in Table 13.

Table 12. Computational form for a Case III covariance analysis for the data of Table 10.

Treatment	Replicate I		Replicate II		Sums		Means $\bar{y}$	Means $\bar{x}$
1	$\bar{y}_{11.}$ =102.0	$\bar{x}_{11.}$ =15.0	$\bar{y}_{12.}$ = 75.0	$\bar{x}_{12.}$ =10.5	177.0	25.5		
	w_{11} = 3	n_{11} = 1	w_{12} = 5	n_{12} = 2	8	3		
	$w_{11}\bar{y}_{11.}$=306.0	$w_{11}\bar{x}_{11.}$=45.0	$w_{12}\bar{y}_{12.}$=375.0	$w_{12}\bar{x}_{12.}$=52.5	681.0	97.5	85.1250	12.1875
2	$\bar{y}_{21.}$ = 82.5	$\bar{x}_{21.}$ = 8.0	$\bar{y}_{22.}$ = 76.0	$\bar{x}_{22.}$ =14.0	158.5	22.0		
	w_{21} = 5	n_{21} = 2	w_{22} = 3	n_{22} = 1	8	3		
	$w_{21}\bar{y}_{21.}$=412.5	$w_{21}\bar{x}_{21.}$=40.0	$w_{22}\bar{y}_{22.}$=228.0	$w_{22}\bar{x}_{22.}$=42.0	640.5	82.0	80.0625	10.2500
3	$\bar{y}_{31.}$ = 75.0	$\bar{x}_{31.}$ = 4.5	$\bar{y}_{32.}$ = 74.0	$\bar{x}_{32.}$ = 2.0	149.0	6.5		
	w_{31} = 5	n_{31} = 2	w_{32} = 3	n_{32} = 1	8	3		
	$w_{31}\bar{y}_{31.}$=375.0	$w_{31}\bar{x}_{31.}$=22.5	$w_{32}\bar{y}_{32.}$=222.0	$w_{32}\bar{x}_{32.}$= 6.0	597.0	28.5	74.6250	3.5625
4	$\bar{y}_{41.}$ = 71.0	$\bar{x}_{41.}$ =11.0	$\bar{y}_{42.}$ = 57.0	$\bar{x}_{42.}$ = 4.5	128.0	15.5		
	w_{41} = 3	n_{41} = 1	w_{42} = 5	n_{42} = 2	8	3		
	$w_{41}\bar{y}_{41.}$=213.0	$w_{41}\bar{x}_{41.}$=33.0	$w_{42}\bar{y}_{42.}$=285.0	$w_{42}\bar{x}_{42.}$=22.5	498.0	55.5	62.2500	6.9375
5	$\bar{y}_{51.}$ = 53.0	$\bar{x}_{51.}$ = 2.0	$\bar{y}_{52.}$ = 62.0	$\bar{x}_{52.}$ = 7.5	115.0	9.5		
	w_{51} = 3	n_{51} = 1	w_{52} = 5	n_{52} = 2	8	3		
	$w_{51}\bar{y}_{51.}$=159.0	$w_{51}\bar{x}_{51.}$= 6.0	$w_{52}\bar{y}_{52.}$=310.0	$w_{52}\bar{x}_{52.}$=37.5	469.0	43.5	58.6250	5.4375
Sums	383.5	40.5	344.0	38.5	727.5	79.0		
	19	7	21	8	40	15		
	1465.5	146.5	1420.0	160.5	2885.5	307.0	72.1375	7.6750

Table 13. Covariance analysis for the data in Table 12.

Source of variation	d.f.	y^2	xy	x^2
		Sum of products		
Total (on means)	9	6111.49	1273.29	648.78
Replicate (ign. treat.)	1	902.62	6.42	.05
Treatment (elim. rep.) = T	4	3826.62	873.68	397.10
Error = E	4	1382.25	393.19	251.63
Within subclasses = S	5	238.50	6.00	4.00
Treatment (ign. rep.)	4	4144.15	850.48	395.65

Source of variation	Regression d.f.	Regression ss	Adjusted sum of squares d.f.	Adjusted sum of squares ss	Adjusted sum of squares ms
S	1	9.00	4	229.50	57.38
E	1	614.39	3	767.86	255.95
E + T	1	2474.00	7	2734.87	--
Treatment (elim. other effects)			4	1967.01	491.75

$\mu' = 72.3350$ $\quad \tau_4' = -9.0975$ $\quad \tau_{x1}' = 4.5525$ $\quad \tau_{x5}' = -2.1975$

$\tau_1' = 13.7775$ $\quad \tau_5' = -12.7225$ $\quad \tau_{x2}' = 2.5150$ $\quad \mu_x' = 7.6850$

$\tau_2' = 6.7400$ $\quad \rho_1' = 3.9500$ $\quad \tau_{x3}' = -4.1725$ $\quad \rho_{x1}' = 0.2000$

$\tau_3' = 1.3025$ $\quad \rho_2' = -3.9500$ $\quad \tau_{x4}' = -0.6975$ $\quad \rho_{x2}' = -0.2000$

$\hat{\beta} = 1.5000$ $\qquad \hat{\mu} + \hat{\tau}_3 = 80.0245$

$\hat{\beta}_1 = 1.5626$ $\qquad \hat{\mu} + \hat{\tau}_4 = 64.1946$

$\hat{\mu} + \hat{\tau}_1 = 78.8661$ $\qquad \hat{\mu} + \hat{\tau}_5 = 62.9134$

$\hat{\mu} + \hat{\tau}_2 = 75.0123$ $\qquad \hat{\rho}_1 = 3.6375 = -\hat{\rho}_2$.

A second estimate of $\sigma^2_{\rho\tau}$ is obtained from formula (III-31) as:

$$\hat{\sigma}^2_{\rho\tau} = \frac{1}{1.014}\left\{\frac{767.86}{27.5 + 2.6875 + 13.4375 - 5.3797} - 57.38\right\} = 10.01 \ .$$

Since the second estimate of $\sigma^2_{\rho\tau}$ is almost identical with the first estimate of $\sigma^2_{\rho\tau}$, which is $\hat{\sigma}^2_{\rho\tau} = 10.73$, the second set of weights would be almost identical with the first. Likewise, the new analysis of variance table and the adjusted effect differences would be very similar. Therefore, the iterative analysis of variance stops here and the adjusted treatment means are obtained (Table 13). Under the assumption of fixed weights the tests of significance suggested previously may be carried out using the interaction mean square as the error mean square.

After obtaining $\hat{\beta}_1 = 393.19/251.63 = 1.562572$, the $\hat{\mu} + \hat{\tau}_i$ and $\hat{\rho}_j$ values may be computed. The two equations for the $\hat{\rho}_j$'s are (III-16):

$$\frac{75}{8}\hat{\rho}_1 - \frac{75}{8}\hat{\rho}_2 = 1465.5 - \frac{3}{8}(681.0 + 498.0 + 469.0) - \frac{5}{8}(640.5 + 597.0)$$

$$- 1.562572\left[146.5 - \frac{3}{8}(97.5 + 55.5 + 43.5) - \frac{5}{8}(82.0 + 28.5)\right]$$

$$= 74.0625 - 5.859645 = 66.202855 \ ;$$

$$\hat{\rho}_1 = - \hat{\rho}_2 \ .$$

Alternatively, the two equations involving $\hat{\rho}_1$ and $\hat{\rho}_2$ from formula (III-13) are:

$$\hat{\rho}_1 \left[\frac{75}{8} - \frac{1}{253.13}(146.5 - \frac{3}{8}(196.5) - \frac{5}{8}(110.5))^2\right]$$

$$- \hat{\rho}_2 \left(\frac{75}{8} + \frac{1}{253.13}(3.75)(-3.75)\right)$$

$$= 74.0625 - \frac{422.81}{253.13}(3.75) = 67.798772 \ ;$$

and

$$\hat{\rho}_1 = - \hat{\rho}_2 \ ,$$

where $253.13 = 648.78 - 395.65$ and $422.81 = 1273.29 - 850.48$.

Summary

Variance and covariance analyses have been classified under three categories, *viz*., Case I, interaction absent; Case II, interaction present and the effects assumed to be fixed effects; and Case III, interaction present and the interaction effects and at least one of the main effects of the factors represented in the interactions assumed to be random effects. The statistical procedures for the three cases have been derived for two-way and three-way classifications and are illustrated with numerical examples for the two-way classification with a covariate. The procedures for a q-way classification with b covariates are indicated. In addition, the restrictions on the linear model are discussed.

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